Module 8

Outline: DAC

- Sample and Hold (S/H)
- Reconstruction - Sect. 9.3.1
- Digital Up-Sampling - Sect. 10.3
- Efficient Upsampling - Sect. 10.5.2
• See Fig. 9.22 on pg. 767 => S/H reconstruction

• Digital Upsampling

\[ X[n] \xrightarrow{\text{System}} Y[n] \]

\[ = x_a \left( \frac{n}{L F_s} \right) \]

• where \( L = \text{integer} \)

• i.e., increasing sampling by \( L \) just prior to S/H
Preamble:

\[ x[n] \ast \delta[n-n_0] = x[n-n_0] \]

In particular, if \( n_0 = 0 \):

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \]

Consider:

\[ \sum_{k=-\infty}^{\infty} x[k] \delta[n-2k] \]
\[
\begin{align*}
\{ x[0], x[1], & \ldots, x[n-2], x[n-1] \} \\
= & \{ 0, x[-1], x[0], x[1], x[2], \ldots, x[n-2], x[n-1] \} \\
& + \{ x[2], x[3], x[4], \ldots, x[n-1] \} \\
& + \{ x[n-1] \}
\end{align*}
\]
more generally:

insert \( L-1 \) zeroes between each pair of samples

\[
x[n] \xrightarrow{\uparrow L} \sum_{k=-\infty}^{8} x[k] \delta[n-kl]
\]

End of Preamble
Recall: \[ X_a(F) = \mathcal{F}\{x_a(t)\} \]

DTFT \( x_a(n) \):

\[ \text{DTFT} \left\{ x_a\left(\frac{n}{F_s}\right) \right\} \]

\[ \omega = 2\pi \frac{F}{F_s} \]

DTFT \( x_a\left(\frac{n}{2F_s}\right) \):

\[ \text{DTFT} \left\{ x_a\left(\frac{n}{2F_s}\right) \right\} \]

\[ \omega = 2\pi \frac{2F_s}{2F_s} \]

\[ L = 2 \]
- as long as $F_s > 2W$
  (i.e., $|X_a(F)| \approx 0$ for $|F| > W$)
- then $X_a(t)$ may be reconstructed

$$X_a(t) = \sum_{k=-\infty}^{\infty} x[k] h_{LP}(t-kT_s)$$
\[ y[n] = X_a\left(\frac{n}{L}\right) = X_a\left(\frac{n}{LF_s}\right) \]

\[
= \sum_{k=-\infty}^{\infty} x[k] h_{LP}\left(\frac{n}{L} - kT_s\right) \]

\[
= \sum_{k=-\infty}^{\infty} x[k] h_{LP}[n - kL]\]

where: \[ h_{LP}[n] = h_{LP}\left(\frac{n}{L}\right) = h_{LP}\left(\frac{n}{LF_s}\right) \]
\[
\begin{align*}
\{ s[n] \} = \mathcal{F}\{ x[n] \} \\
\{ x[n] \} = x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \\
\{ y[n] \} = \{ [n] \} \ast [n] \\
\{ n-k\} \rightarrow h[n] = \sum_{k=-\infty}^{\infty} h[k] \delta[n-k] \\
\{ h[n] \} \ast \{ n-k\} = h[n] \ast [n] \\
\{ y[n] \} = \{ h[n] \} \ast \{ n-k\} \\
\{ n-k\} \rightarrow \text{convolution is distributive} \\
\end{align*}
\]
Spec's on the digital LPF:

\[ H_{LP}(\omega) = \begin{cases} 1 & \frac{2\pi W}{LF_s} < |\omega| < \frac{2\pi(F_s - W)}{LF_s} \\ \text{otherwise} & \end{cases} \]
Example: for speech with bandwidth $\approx 4$ kHz sampled at $F_s = 12$ KHz, desire to increase sampling rate by $L = 2$ to 24 KHz.

\[
\omega_p = 2\pi \frac{4}{2(12)} = \frac{\pi}{3}
\]
\[
\omega_s = 2\pi \frac{(12-4)}{2(12)} = \frac{2\pi}{3}
\]

See demo upsampling 2 (eg1.m)
Efficient Digital Upsampling

$L = 2$ for illustration

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h_{L,2}[n-2k] \]

- Consider \( y[2n] \) and \( y[2n+1] \) separately
\[ y_0[n] = y[2n] \]
\[ = \sum_{k=-\infty}^{\infty} x[k] h_{LP}[2n - 2k] \]
\[ = x[n] * h_0[n] \]

where:
\[ h_0[n] = h_{LP}[2n] \]

\[ x[n] \rightarrow h_0[n] = h_{LP}[2n] \rightarrow y_0[n] = y[2n] \]
\[ y_1[n] = y[1 + 2n] \]
\[ = \sum x[k] h_{LP}[1 + 2n - 2k] \]
\[ = x[n] * h_1[n] \]

where: \( h_1[n] = h_{LP}[1 + 2n] \)
"commutator" operates at twice original sampling rate
efficient implementation for general case of up-sampling by $L$

$h_0[n] = h_{LP}[ln]

h_1[n] = h_{LP}[1+ln] 

X[n] 

\vdots

h_{L-1}[n] = h_{LP}[L-1+ln]

y[n]

y[Ln]

y[1+Ln]

y[L-1+Ln]

\vdots

commutators operates at $L\cdot F_s$