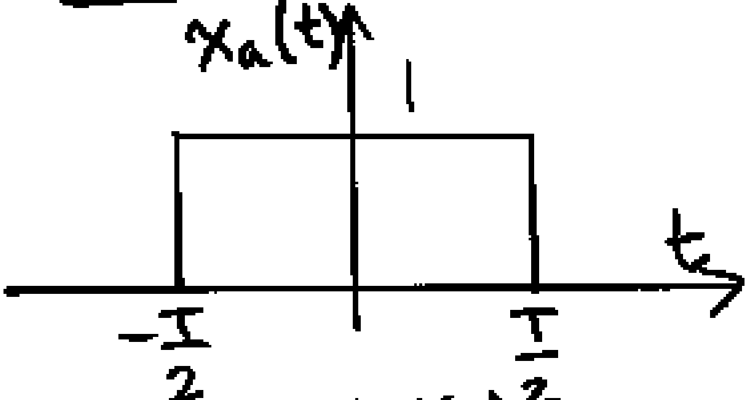
Supplemental Notes

Module 7a

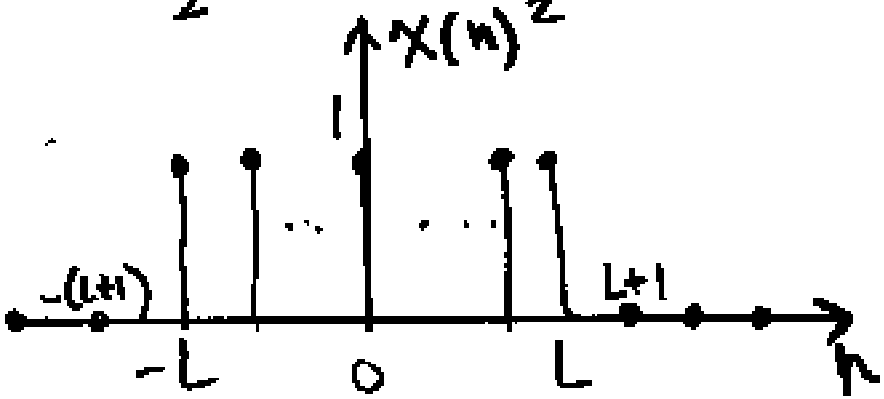
Example

$$x_a(t) = \mathrm{rect}\left(\frac{t}{T}\right)$$

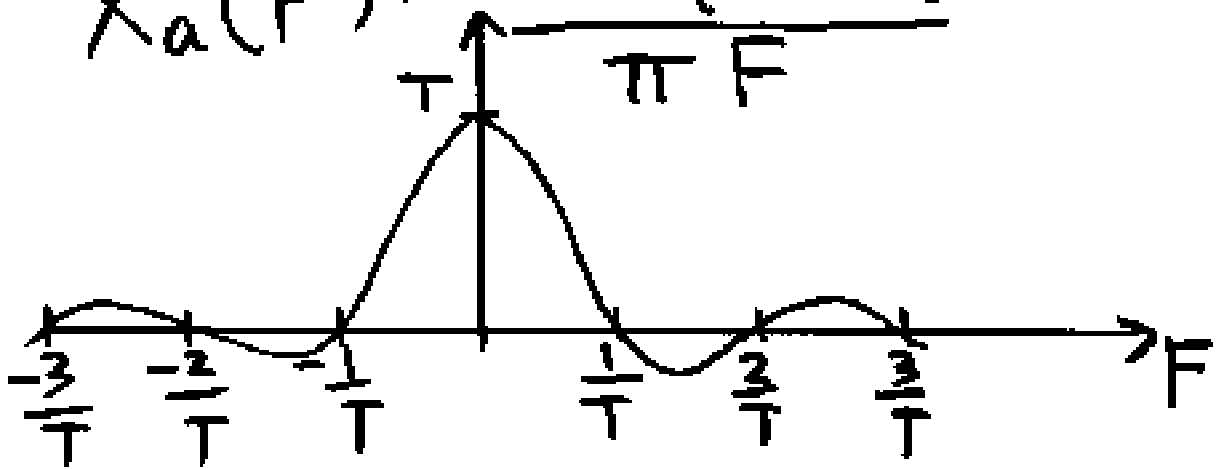


$$x(n) = x_a\left(\frac{n}{F_s}\right)$$

$$F_s = \frac{2L+1}{T}$$



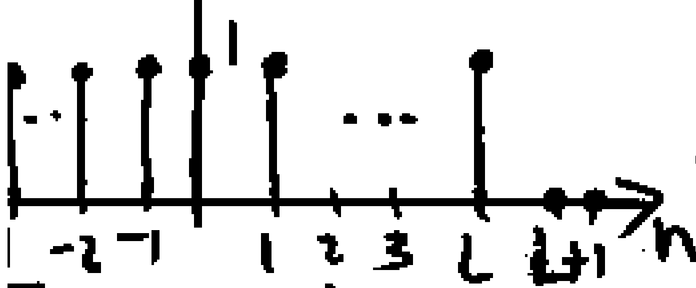
$$X_a(F) = \frac{\sin(\pi F T)}{\pi F}$$



- not strictly bandlimited
- will always be aliasing
- depending on F_s, might pass thru analog LPF with cut-off at $\frac{F_s}{2}$ prior to sampling

· Some DTFT Pairs

$$x(n) \xleftrightarrow{DTFT} X(\omega)$$



$$x(n) = u(n+L) - u(n-(L+1))$$

change of variables $n' = n + L$ $(n = n' - L)$

$$X(\omega) = \sum_{n=-L}^{L} e^{-j\omega n}$$

$$n' \Big|_{0}^{2L}$$

$$= \sum_{n'=0}^{2L} e^{-j\omega(n'-L)}$$

$$X(\omega) = \left\{ \frac{1 - e^{-j\omega(2L+1)}}{1 - e^{-j\omega}} \right\} e^{j\omega L} \Rightarrow \frac{e^{-j\omega\frac{(2L+1)}{2}}}{e^{-j\frac{\omega}{2}}}$$

$$= \frac{\sin\left(\frac{2L+1}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

• recall: $x_a(t) = \text{rect}\left(\frac{t}{T}\right)$

view $x(n)$ as samples of $x_a(t)$

$x(n) = x_a(t)\Big|_{t=\frac{n}{f_s}}$ $\quad f_s = \frac{2L+1}{T}$

relationship between CTFT and DTFT dictates

$$X(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a\left(\frac{F_s}{2\pi}(\omega + k2\pi)\right)$$

$$= \frac{2L+1}{\cancel{T}} \sum_{k=-\infty}^{\infty} \frac{\sin\left(\cancel{\pi}\frac{2L+1}{\cancel{T}}\frac{1}{2\cancel{\pi}}(\omega+k2\pi)\cancel{T}\right)}{\frac{1}{2\cancel{\pi}}\cancel{\pi}\,\frac{2L+1}{\cancel{T}}(\omega+k2\pi)}$$

$$= \sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{2L+1}{2}(\omega+k2\pi)\right)}{\frac{1}{2}(\omega+k2\pi)}$$

$$X(\omega) = \sin\left(\frac{2L+1}{2}\omega\right) \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\left(\frac{\omega + k2\pi}{2}\right)}$$

- Compare with computing DTFT directly:

$$X(\omega) = \sum_{n=-L}^{L} e^{-j\omega n}$$

$$= \frac{\sin\left(\frac{2L+1}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

- one case where we can perfectly quantify aliasing

- look at $k=0$ term in CTFT
- DTFT relationship

$$X(\omega) \approx \frac{\sin\left(\frac{2L+1}{2}\omega\right)}{\frac{\omega}{2}}$$ for $\omega \ll 1$ or $\omega \to 0$

$$= F_s X_a(F) \Big|_{\left(\omega = 2\pi \frac{F}{F_s}\right)} \Rightarrow F = \frac{\omega F_s}{2\pi}$$

- so for small ω, the effect of aliasing is negligible

$$X(\omega) = \frac{\sin\left(\frac{2L+1}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

- for $\omega \ll 1$:

$$\sin\left(\frac{\omega}{2}\right) \approx \frac{\omega}{2}$$

Example 2. $x_a(t) = e^{-bt}u(t)$

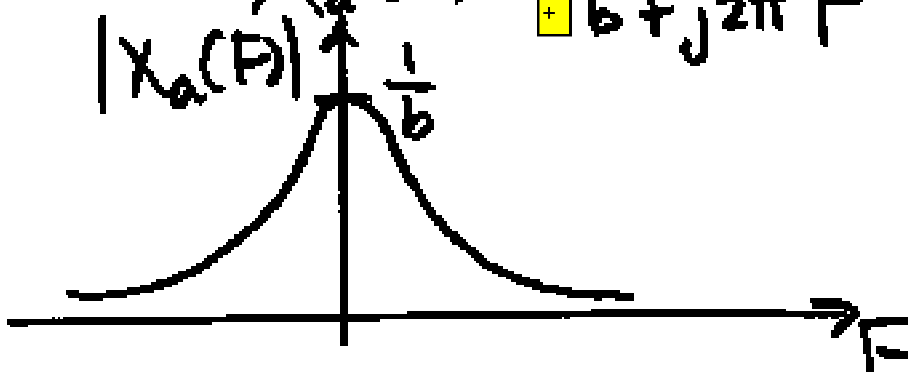
$b > 0$

$x(n) = x_a(nT_s)$

$= e^{-bnT_s}u(nT_s) = a^n u(n)$

where $a = e^{-bT_s}$

$$X_a(F) = \frac{1}{b + j2\pi F}$$



- Relationship between CTFT
- DTFT dictates

$$X(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \frac{1}{b + j\frac{1}{T_s}(\omega + k2\pi)}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{bT_s + j(\omega + k2\pi)} \qquad a = e^{-bT_s}$$

- Compare with computing DTFT directly

assuming $a < 1$:

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$

• for small ω, $e^{-j\omega} \approx 1 - j\omega$

$$X(\omega) \approx \frac{1}{1 - a + aj\omega} = \frac{1}{1 - e^{-bT_s} + j\omega e^{-bT_s}}$$

for $T_s \ll 1$, $e^{-bT_s} \approx 1 - bT_s$

$$X(\omega) \approx \frac{1}{bT_s + j\omega}$$

which is exactly equal to $k=0$ term in DTFT-CTFT relationship

See DTFT pairs on pg. 305 Table 4.6