EE538 DSP I

Module 6

Outline

- Continuous Time Fourier Transform (CTFT) - Sects. 4.1.3-4.1.4
- Relationship CTFT and DTFT – Sect. 4.2.9
- Examples
- See aliaseg.m at web site

- $x_a(t) \xleftrightarrow{\text{CTFT}} X_a(F)$

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi F t}\, dt$$

- or can be expressed in terms of $\Omega = 2\pi F$ (angular frequency)

- Inverse CTFT:

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F t}\, dF$$

$$\int_{F_1}^{F_2} |X_a(F)|^2 dF = \text{energy of } x_a(t) \text{ in the frequency band } F_1 < F < F_2$$

- where: $|X_a(F)|^2 = X_a(F) X_a^*(F)$

$$= X_R^2(F) + X_I^2(F)$$

- where: $X_a(F) = X_R(F) + jX_I(F)$

$$\angle X_a(F) = \tan^{-1}\left\{\frac{X_I(F)}{X_R(F)}\right\}$$

• Recall: Inverse DTFT (IDTFT)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

• where: $X(\omega)$ is the DTFT of $x[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$\equiv X(z)\Big|_{z=e^{j\omega}},\ -\pi < \omega < \pi$$

• $X(\omega)$: periodic with period 2π

- Suppose: $x[n] = x_a(t)\big|_{t=nT_s}$

$$= x_a\left(\frac{n}{F_s}\right)$$

- where: $F_s = \frac{1}{T_s}$ = sampling rate

- How is $X(\omega)$ related to $X_a(F)$?

- sample ICTFT

$$x_a(t)\big|_{t=nT_s} = \int_{-\infty}^{\infty} X_a(F)\, e^{j2\pi F n T_s}\, dF$$

$$= x[n]$$

- change of variables:

$$\omega = 2\pi \frac{F}{F_s} \qquad d\omega = \frac{2\pi}{F_s} dF$$

$$F = \frac{\omega F_s}{2\pi} \qquad dF = \frac{F_s}{2\pi} d\omega$$

$$F \Big|_{-\infty}^{\infty} \Rightarrow \omega \Big|_{-\infty}^{\infty}$$

$$x[n] = \int_{-\infty}^{\infty} X_a\left(\frac{\omega F_s}{2\pi}\right) e^{j\omega n} \frac{F_s}{2\pi} d\omega$$

- evaluate integral via Riemann Sum over disjoint intervals of width 2π centered at

$$\ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$$

$$x[n] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{k2\pi-\pi}^{k2\pi+\pi} F_s X_a\left(\frac{\omega F_s}{2\pi}\right) e^{j\omega n} \, d\omega$$

- another change of variables:

$$\omega' = \omega - k2\pi \qquad \omega' \Big|_{-\pi}^{\pi}$$

$$d\omega' = d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_s \sum_{k=-\infty}^{\infty} X_a\left(\frac{F_s(\omega + k2\pi)}{2\pi}\right) \cdot e^{j(\omega' + k2\pi)} \, d\omega'$$

• Substitute: $\boxed{\omega = \omega' + k2\pi}$

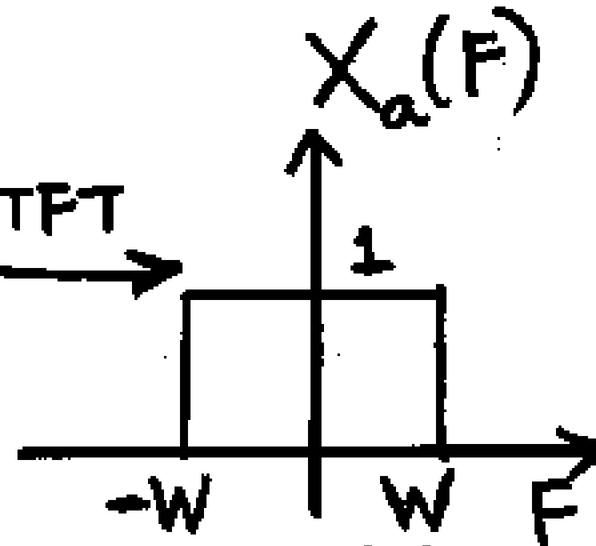
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_s \sum_{k=-\infty}^{\infty} X_a\left(\frac{F_s}{2\pi}(\omega + k2\pi)\right) \cdot e^{j\omega n} \, d\omega$$

$$\underset{\text{DTFT}}{I} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$X(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a\left(\frac{F_s}{2\pi}(\omega + k2\pi)\right)$$

- Example 1.

$$x_a(t) = \frac{\sin(2\pi W t)}{\pi t} \xleftrightarrow{\text{CTFT}}$$



$$X_a(F) = \text{rect}\left(\frac{F}{2W}\right)$$

- where: $\text{rect}(x) = \begin{cases} 1, & |x| < 1/2 \\ 0, & |x| > 1/2 \end{cases}$

$$x[n] = x_a\left(\frac{n}{F_s}\right) = \frac{\sin\left(2\pi \frac{W}{F_s} n\right)}{\pi \, n/F_s}$$

- $X(\omega) = \text{DTFT}\{x[n]\} = \,?$

$$X(\omega) = F_s \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{F_s}{2\pi} \frac{(\omega + k2\pi)}{2W}\right)$$

- consider $k=0$ term above:

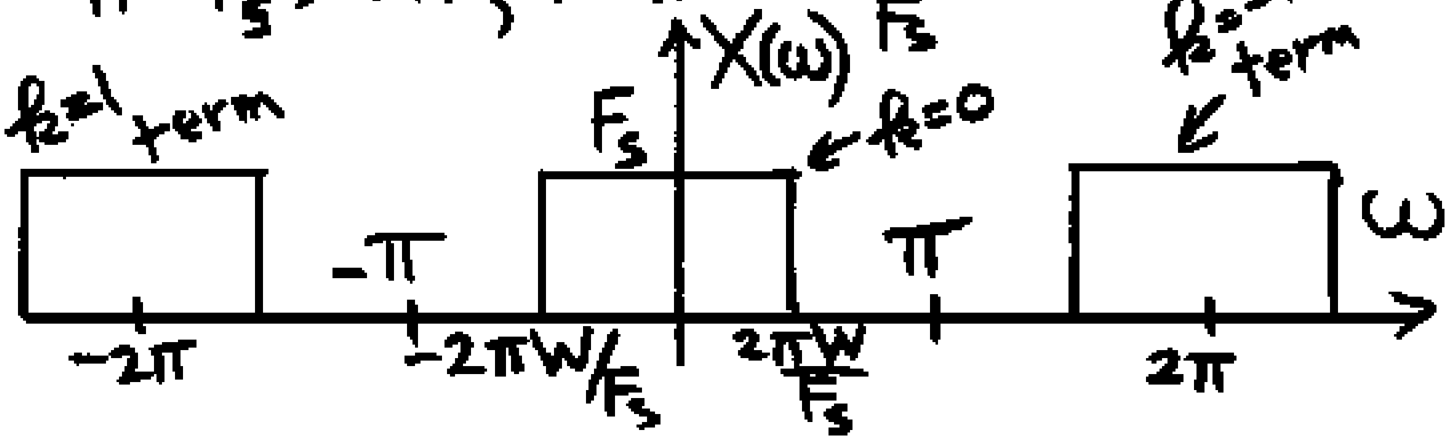
$$\text{rect}\left(\frac{F_s\,\omega}{4\pi W}\right)$$

$$= 1, \text{ when } -\frac{1}{2} < \frac{\omega F_s}{4\pi W} < \frac{1}{2}$$

$$-2\pi W < \omega F_s < 2\pi W$$

$$-2\pi \frac{W}{F_s} < \omega < 2\pi \frac{W}{F_s}$$

- if $F_s > 2W$, then $2\pi \frac{W}{F_s} < \pi$



$$\frac{\sin\left(2\pi \frac{W}{F_s} n\right)}{\pi n} \overset{DTFT}{\longleftrightarrow} \operatorname{rect}\left(\frac{\omega}{\frac{4\pi W}{F_s}}\right)$$

periodic with period 2π

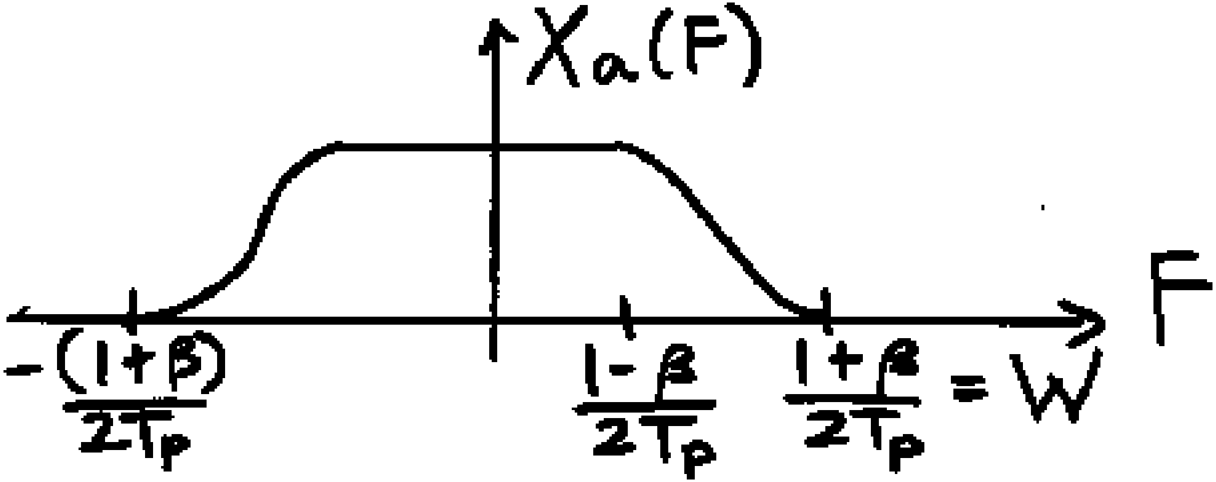
- Verify:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n}\, d\omega$$

- Example 2.

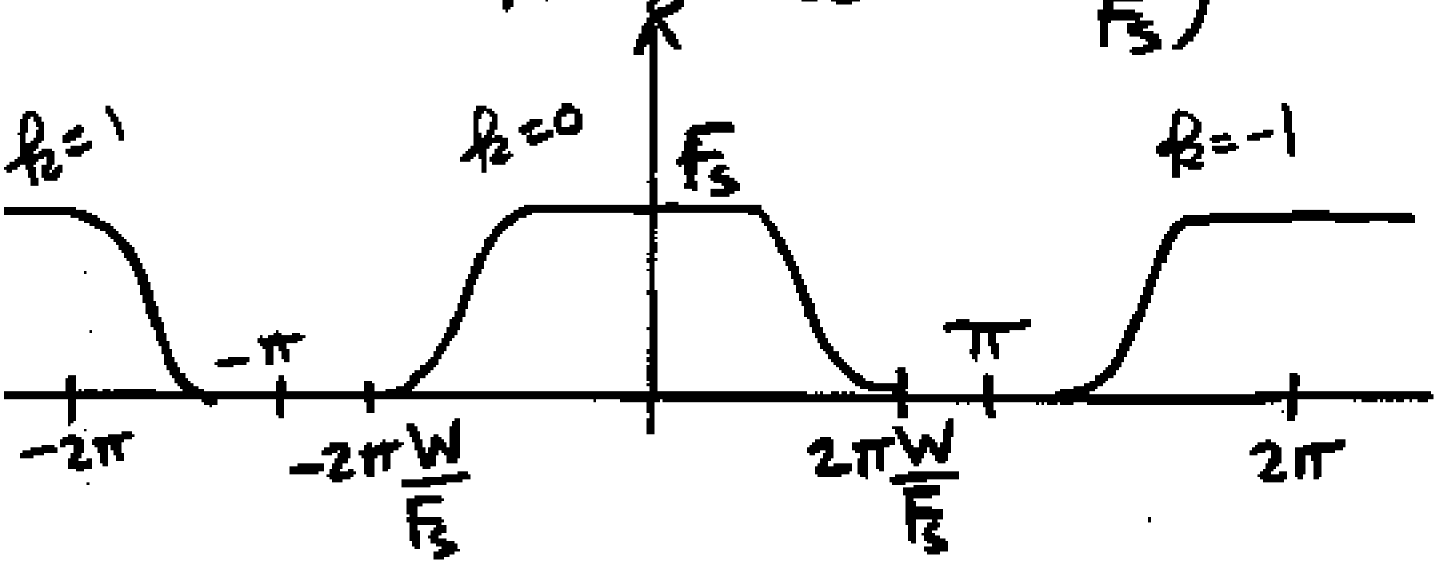
$$x_a(t) = \frac{\sin\left(\pi \frac{t}{T_p}\right)}{\pi t / T_p} \cdot \frac{\cos\left(\pi \beta \frac{t}{T_p}\right)}{1 - 4\beta^2 t^2 / T_p^2}$$

- $0 < \beta < 1$ for digital communications

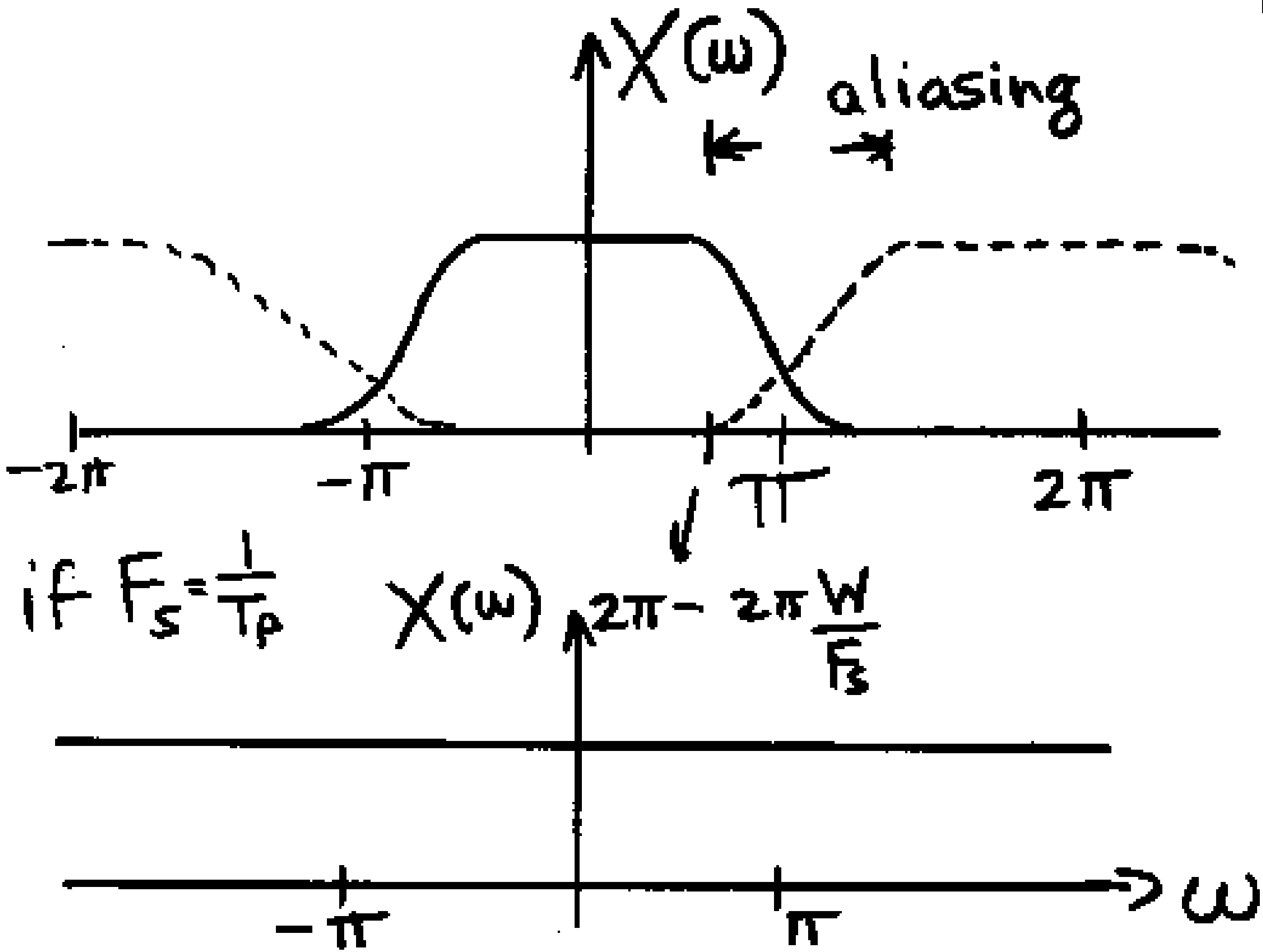


• if $F_s > 2W = 2\left(\frac{1+\beta}{2T_p}\right) = \frac{1+\beta}{T_p}$

(recall: mapping $\omega = 2\pi \frac{F}{F_s}$)



• if $F_s < 2W \Rightarrow$ aliasing


X(ω)
aliasing
−2π
−π
π
2π
if $F_s = \frac{1}{T_p}$
X(ω)
$2\pi - 2\pi \frac{W}{F_s}$
−π
π
ω

- in particular, with $0 < \beta < 1$

$$F_s = \frac{1}{T_p} < 2\left(\frac{1+\beta}{2T_p}\right) = \frac{1+\beta}{T_p}$$

$$x_a(nT_p) = \underbrace{\frac{\sin(n\pi)}{n\pi}}_{\delta[n]} \cdot \frac{\cos(\pi\beta n)}{1-(2\beta n\pi)^2}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-jn\omega} = 1 \cdot e^{j0} = 1$$

for all ω

$$\delta[n] \xleftrightarrow{\text{DTFT}} 1$$