EE538 DSP I

Module 4

Outline:

- Z-Transform (cont.)
- Sects. 3.4.3 and Sect. 3.6.4

- Relationship between ZT and DTFT

Sect. 4.2.6, 4.4.2

- Example.

$$y[n] = \frac{13}{4} y[n-1] - \frac{3}{4} y[n-2] + x[n]$$

- Determine all possible impulse responses

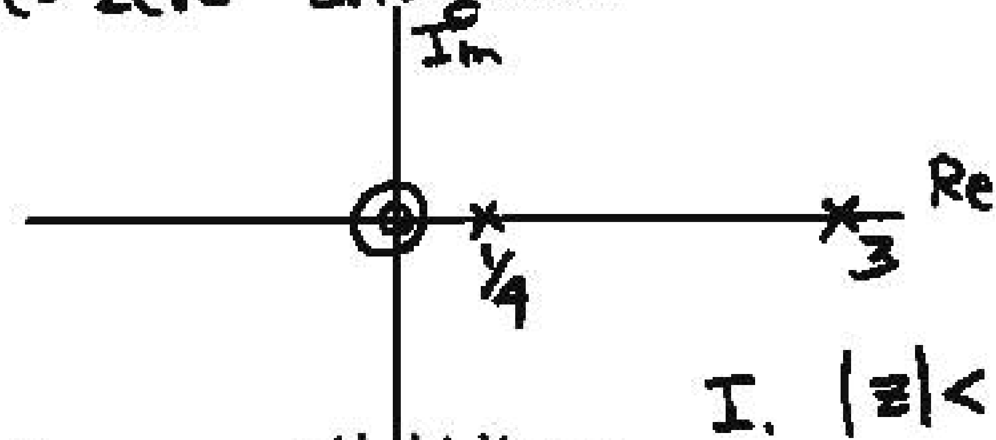
$$Y(z) = \frac{13}{4} z^{-1} Y(z) - \frac{3}{4} z^{-2} Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{13}{4} z^{-1} + \frac{3}{4} z^{-2}}$$

$$H(z) = \frac{z^2}{z^2 - \frac{13}{4} z + \frac{3}{4}} =$$

$$H(z) = \frac{z^2}{\left(z-\frac{1}{4}\right)(z-3)}$$

· pole-zero diagram :



Three possibilities for ROC

I. $|z| < \frac{1}{4}$

II. $\frac{1}{4} < |z| < 3$

III. $|z| > 3$

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$$H(z) = A_1 \frac{z}{z-\frac{1}{4}} + A_2 \frac{z}{z-3}$$

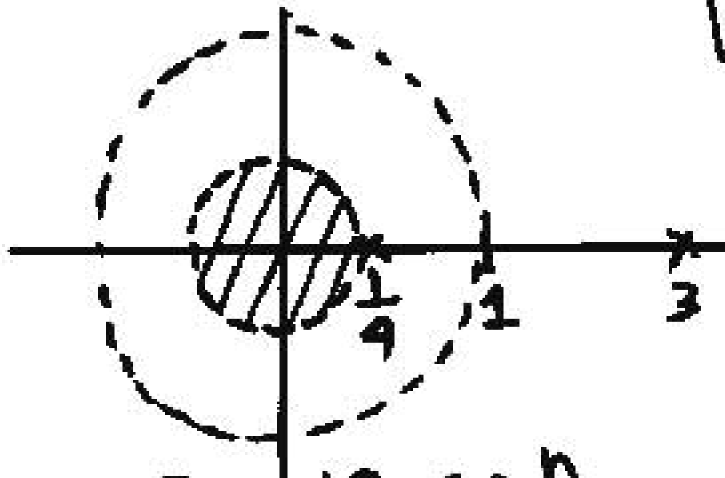
$$A_1 = \frac{z-\frac{1}{4}}{z} H(z) \Big|_{z=\frac{1}{4}} =$$

$$= \frac{z}{z-3} \Big|_{z=\frac{1}{4}} = -\frac{1}{11}$$

$$A_2 = \frac{12}{11}$$

$$H(z) = -\frac{1}{11} \frac{z}{z-\frac{1}{4}} + \frac{12}{11} \frac{z}{z-3}$$

I. $|z| < \frac{1}{4}$

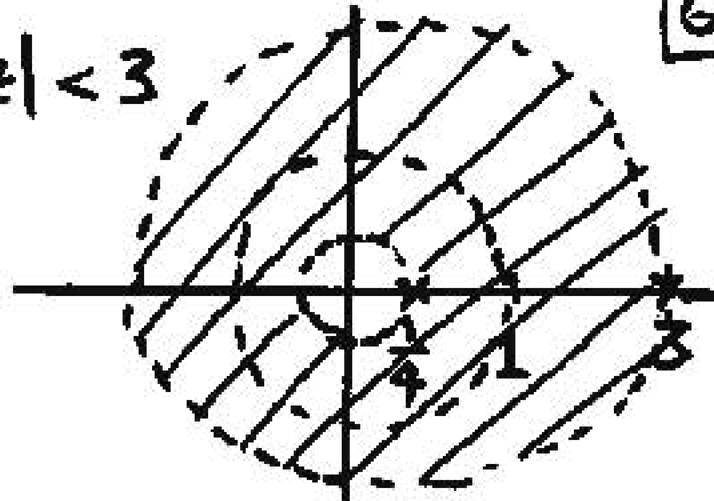


$$h[n] = \frac{1}{11}\left(\frac{1}{4}\right)^n u[-n-1] - \frac{12}{11}(3)^n u[-n-1]$$

- Causal? No! Left-sided sequence
- BIBO Stability? No!

$$\lim_{n \to -\infty} |h[n]| = +\infty$$

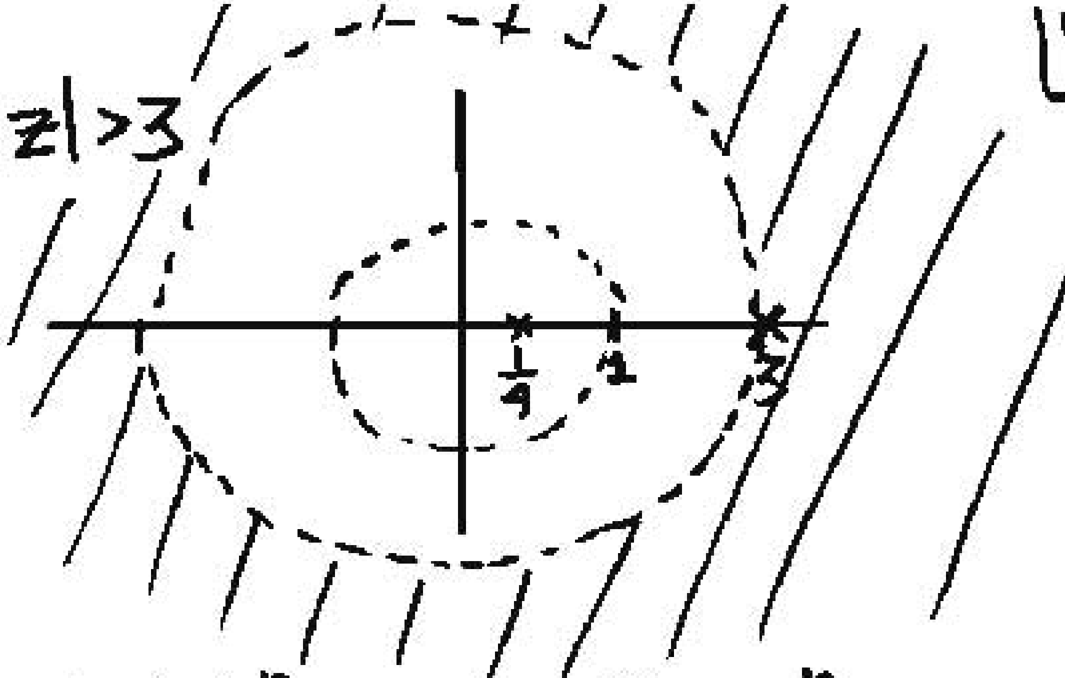
II. ROC: $\frac{1}{4} < |z| < 3$



$$h[n] = -\frac{1}{11}\left(\frac{1}{4}\right)^n u[n] - \frac{12}{11}(3)^n u[-n-1]$$

- Causal? No! Two-Sided Sequence
- Stability? Yes! $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

III. $|z|>3$



$$h[n] = -\frac{1}{11}\left(\frac{1}{4}\right)^n u[n] + \frac{12}{11}(3)^n u[n]$$

- Causal? Yes!
- Stable? No!

- each of these three sol'ns. for the impulse response satisfy

$$h[n] = \frac{13}{4} h[n-1] - \frac{3}{4} h[n-2] + \delta[n]$$

- only the causal sol'n. has practical value
- End of Example

- **Stability**: requires ROC to include unit circle, $|z| = 1$
- Supporting argument:
- invoke triangle inequality ($|a+b| \leq |a| + |b|$)
- $|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n] z^{-n}|$
- on unit circle: $|z^{-n}| = |z|^{-n} = 1^{n} = 1$

- on unit circle:

$$\underbrace{|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n]| < \infty}_{\text{for BIBO Stability}}$$

- ROC must include unit circle for BIBO Stability

• Stability and Causality

• requires $|z| > |p_N|$ must include unit circle, $|z| = 1$

$\Rightarrow |p_N| < 1$

$\Rightarrow$ all poles must be located within unit circle

• note: for distinct poles:

$$h[n] = \sum_{k=1}^{N} A_k p_k^n u[n]$$

- Onto Chap. 4 on DTFT

Discrete-Time Fourier Transform

- Recall:

$$x[n] = z_0^n \longrightarrow \boxed{\begin{array}{c} \text{LTI} \\ h[n] \end{array}} \longrightarrow y[n] = H(z_0)\, z_0^n$$

$\forall n$

- Consider z_0 on the unit circle
- recall: $e^{j\theta} = \cos\theta + j\sin\theta$
- $z_0 = e^{j0} = 1$; $z_0 = e^{j\pi} = -1 = e^{-j\pi}$
- $z_0 = e^{j\frac{\pi}{2}} = j$; $z_0 = e^{-j\frac{\pi}{2}} = -j$

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n]\end{matrix}} \longrightarrow y[n] = H(\omega_0)\, e^{j\omega_0 n}$$

- where: $H(\omega) \underbrace{=}_{\text{notational problem}} H(z)\Big|_{z=e^{j\omega}}$

- $H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$ $\Big\}$ DTFT of $h[n]$

- only defined if ROC includes $|z|=1$

  $\Rightarrow$ only defined for stable systems

$$H(\omega_0) = |H(\omega_0)| e^{j\angle H(\omega_0)}$$

$$x[n] = Ae^{j(\omega_0 n+\theta)} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow |H(\omega_0)|A \cdot e^{j(\omega_0 n+\theta+\angle H(\omega_0))}$$

$-\infty < n < \infty$

- recall:

$$\cos(\omega_0 n) = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$A\cos(\omega_0 n+\theta) \longrightarrow \boxed{h[n]} \longrightarrow A|H(\omega_0)| \cdot \cos(\omega_0 n+\theta+\angle H(\omega_0))$$

- Return to Difference Eqns.

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= b_0 z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

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$$H(\omega) = H(z)\Big|_{z=e^{j\omega}}$$

$$H(\omega) = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

$$|H(\omega)| = |b_0| \prod_{k=1}^{M}|e^{j\omega} - z_k| \Big/ \prod_{k=1}^{N}|e^{j\omega} - p_k|$$

$$\angle H(\omega) = \omega(N-M) + \angle b_0 + \sum_{k=1}^{M} \angle\left(e^{j\omega} - z_k\right) - \sum_{k=1}^{N} \angle\left(e^{j\omega} - p_k\right)$$

- See pg. 323 in Text for vector interpretation