EE538 DSP I

Module 30

Outline

- Minimum Variance Spectral Estimation – Sect. 12.4
  - see demos: MinVarARMA.m
    MinVarSoS.m

- Minimum Variance Spectral Estimation (MV Method)
- <u>approach</u>: pass $x[n]$ thru an FIR filter which ~~minimizes~~ is a measure the mean square value of the ~~filter output~~ spectral energy at ω_0 $\Rightarrow$ do this for all $-\pi < \omega_0 < \pi$
- spectrum is a plot of the resulting energy at each frequency

- criterion for designing FIR filter for a specific frequency ω_0:
- minimize mean square value of the filter output subject to constraint that frequency response at $\omega = \omega_0$ be unity
- ideally, the FIR filter should only pass energy at $\omega = \omega_0$
- repeat this for all frequencies in $-\pi < \omega < \pi$

- plot of mean square value of filter output (different FIR filter for each frequency) output for all ω is the spectral estimate
- Mathematical Development

$$y[n] = \sum_{k=0}^{M-1} h(k;\omega_0)\, x[n-k]$$

- choose: $h[n;\omega_0]$ as solution to constrained optimization problem

$$\min_{h[n;\omega_0]} E\{|y[n]|^2\}$$

subject to:

$$\mathrm{DTFT}\{h[n;\omega_0]\}\Big|_{\omega=\omega_0} = 1$$

- Rewriting:

$$\min_{h[n;\omega_0]} \sum_{k=0}^{M-1}\sum_{l=0}^{M-1} h(k;\omega_0)\,h^*(l;\omega_0)\cdot E\{x[n-k]\,x^*[n-l]\}$$

s.t.: $\displaystyle\sum_{n=0}^{M-1} h[n;\omega_0]e^{-j\omega_0 n} = 1$

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• definitions:

$$\underset{M\times 1}{\underline{h}(\omega_0)} = \left[h[0;\omega_0], h[1;\omega_0], \ldots, h[M-1;\omega_0]\right]^T$$

$$\underset{M\times 1}{\underline{s}(\omega_0)} = \left[1, e^{j\omega_0}, e^{j2\omega_0}, \ldots, e^{j(M-1)\omega_0}\right]^T$$

• constraint can be expressed as

$$\underline{s}^H(\omega_0)\,\underline{h}(\omega_0) = 1$$

$H = *T$ = conjugate transpose — Hermitian transpose

- objective function may be expressed as

$$\underline{h}^H(\omega_0)\,\underline{R}_M\,\underline{h}(\omega_0) \quad \left\{ \begin{array}{l} \text{quadratic} \\ \text{form} \end{array} \right.$$

- where:

$$\underset{\text{Toeplitz-Hermitian}}{\underline{R}_M} = \begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] & \cdots & r_{xx}^*[M-1] \\ r_{xx}[1] & r_{xx}[0] & & \\ \vdots & & \ddots & \\ r_{xx}[M-1] & & & r_{xx}[0] \end{bmatrix}$$

$$\underset{\underline{h}(\omega_0)}{\text{Minimize}} \quad \underline{h}^H(\omega_0)\,\underline{R}_M\,\underline{h}(\omega_0) \quad \text{s.t.} \quad \underline{s}^H(\omega_0)\,\underline{h}(\omega_0) = 1$$

- Solve via method of Lagrange multipliers – converts constrained optimization into unconstrained one

$$\underset{\underline{h}(\omega_0)}{\text{Min}} \quad \underline{h}^H(\omega_0)\,\underline{R}_M\,\underline{h}(\omega_0) + \underbrace{\lambda}_{\text{Lagrange multiplier}}\left[1 - \underline{s}^H(\omega_0)\,\underline{h}(\omega_0)\right]$$

$$\nabla_{\underline{h}} = \left[\frac{\partial}{\partial h[0;\omega_0]}, \ldots, \frac{\partial}{\partial h[M-1;\omega_0]} \right]^T$$

- can show:

$$\nabla_{\underline{h}} \left(\underline{h}^H(\omega_0) \underline{R}\, \underline{h}(\omega_0) \right) = 2\, \underline{R}_M\, \underline{h}(\omega_0)$$

$$\nabla_{\underline{h}} \left(\underline{s}^H(\omega_0)\, \underline{h}(\omega_0) \right) = \underline{s}(\omega_0)$$

(similar to: $\frac{d}{dx}(ax^2) = 2ax$

$\frac{d}{dx}(ax) = a$)

- gradient of augmented objective function is

$$2\,\underline{R}_M\,\underline{h}(\omega_0) - \lambda\,\underline{s}(\omega_0) = \underset{M\times 1}{\underline{0}}$$

$$\underline{h}(\omega_0) = \frac{\lambda}{2}\,\underline{R}_M^{-1}\,\underline{s}(\omega_0)$$

- λ is determined by satisfying the constraint

$$\underline{s}^H(\omega_0)\,\underline{h}(\omega_0) = \frac{\lambda}{2}\,\underline{s}^H(\omega_0)\,\underline{R}_M^{-1}\,\underline{s}(\omega_0) = \underline{1}$$

$$\lambda = \frac{2}{\underline{s}^H(\omega_0)\,\underline{R}_M^{-1}\,\underline{s}(\omega_0)}$$

$$\underline{h}_{opt}(\omega_0) = \frac{\underline{R}_M^{-1}\,\underline{s}(\omega_0)}{\underline{s}^H(\omega_0)\,\underline{R}_M^{-1}\,\underline{s}(\omega_0)}$$

- mean square value of filter output

$$\underline{h}_{opt}^H(\omega_0)\,\underline{R}_M\,\underline{h}_{opt}(\omega_0) = \frac{\underline{s}^H(\omega_0)\,\underline{R}_M^{-1}\,\underline{R}_M\,\underline{R}_M^{-1}\,\underline{s}(\omega_0)}{\left(\underline{s}^H(\omega_0)\,\underline{R}_M^{-1}\,\underline{s}(\omega_0)\right)^2}$$

$$\hat{S}_{xx}^{MV}(\omega) = \frac{1}{\underline{s}^H(\omega)\,\underline{R}_M^{-1}\,\underline{s}(\omega)}$$

$$-\pi < \omega < \pi$$