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EE538

DSP I

Module 3

Outline: Z-Transform

Relevant P & M Sections:

3.1, 3.2, 3.4.3, 3.6.1-3.6.4

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• Example:  $a = \text{constant}$ , initially <sup>system</sup> at rest  
 $y[n] = a y[n-1] + x[n]$

• impulse response:  $x[n] = \delta[n]$   
 $h[n] = a h[n-1] + \delta[n]$

$h[n] = 0$  for  $n < 0 \Rightarrow$  causal system

• for  $n = 0$ :  $\overbrace{h[-1]} = 0$   
 $h[0] = a h[-1] + 1 = 1$

• for  $n > 0$ :  $h[n] = a h[n-1]$   
consider:  $a^n = a a^{n-1}$

$$h[n] = a^n u[n]$$

• input =  $x[n] = u[n]$

output  $y[n] = ? = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$

$$y[n] = h[n] * x[n] = \sum_{k=0}^n a^k u[k] u[n-k]$$

$$= \{a^n u[n]\} * u[n] \quad n \geq 0$$

$$y[n] = \sum_{k=0}^n a^k \quad \left. \vphantom{\sum_{k=0}^n a^k} \right\} \text{ See pg 81}$$

$$y[n-1] = \sum_{k=0}^{n-1} a^k$$

in P4M Text

$$-a(y[n] - y[n-1]) = a^n \quad (1)$$

$$y[n] - ay[n-1] = 1 \quad (2)$$

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$$y[n](1-a) = 1 - a^{n+1} \quad (1) + (2)$$

$$y[n] = \frac{1 - a^{n+1}}{1 - a} = \sum_{k=0}^n a^k$$

$$\text{if } |a| < 1, \quad \lim_{n \rightarrow \infty} \sum_{k=0}^n a^k = \frac{1}{1-a}$$

$$\text{if } |a| > 1 \Rightarrow \infty$$

# Z-Transform

Recall:



$$H_a(s) = \underbrace{\mathcal{L}\{h_a(t)\}}$$

Laplace Transform

Consider:



for all  $n$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

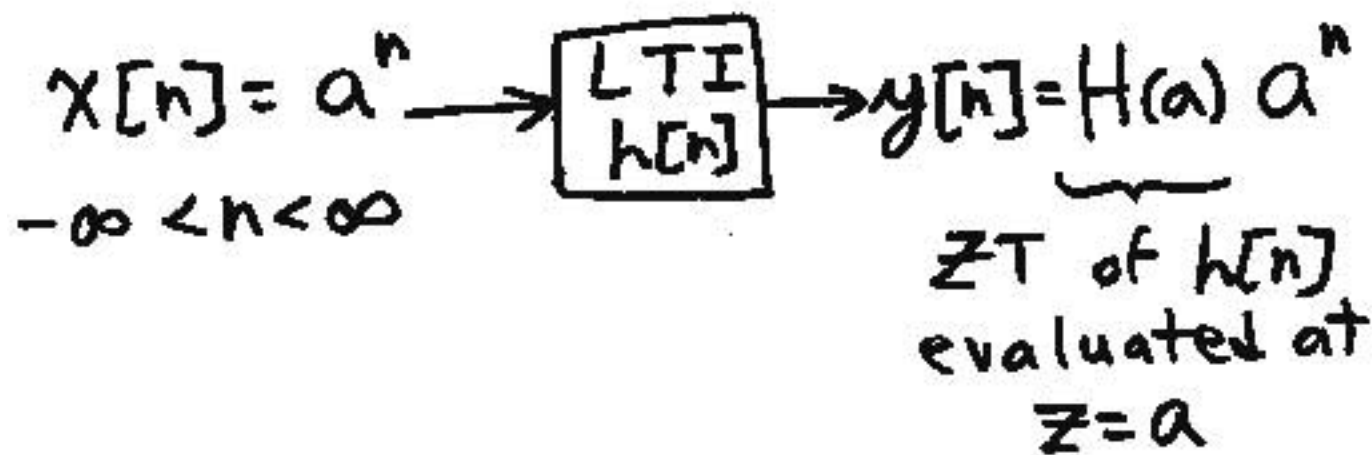
$$= \sum_{k=-\infty}^{\infty} h[k] a^{n-k}$$

$$= \left\{ \sum_{k=-\infty}^{\infty} h[k] a^{-k} \right\} a^n$$

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• Defining Z-Transform (ZT) of  $h[n]$ :

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$



• Examples.

$$1. x[n] = a^n u[n]$$

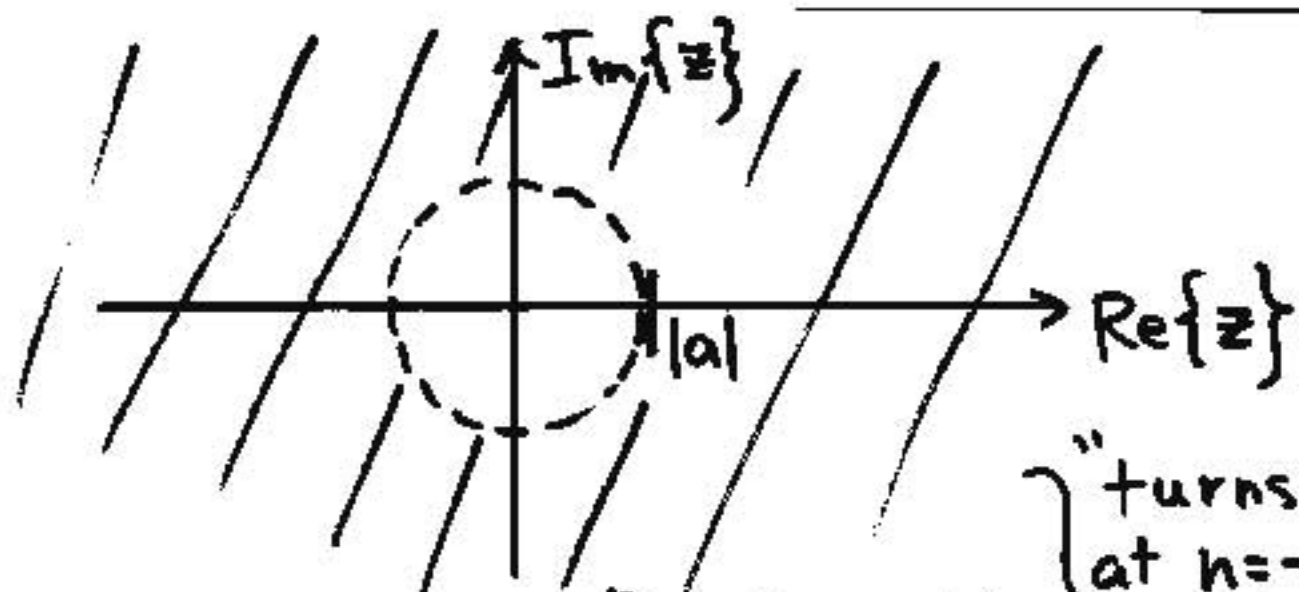
"right-sided"  
sequence

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} \quad |az^{-1}| < 1$$

• Region of Convergence (ROC):  
any value of  $z$  for which  $X(z) < \infty$   
 $\left| \frac{a}{z} \right| < 1 \Rightarrow |a| < |z| \Rightarrow |z| > |a|$





$$2. x[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (az^{-1})^n$$

"turns on"  
at  $n = -\infty$   
and  
"shuts off"  
at  $n = 0$

change of  
variables:  
 $n' = -n$   
( $n = -n'$ )

$$X(z) = \sum_{n'=1}^{\infty} (az^{-1})^{-n'}$$

$$= \sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= \frac{-1}{1 - a^{-1}z} + 1$$

$$= \frac{-a}{a - z} + 1$$

$$= \frac{a}{z - a} + \frac{z - a}{z - a} = \frac{z}{z - a}$$

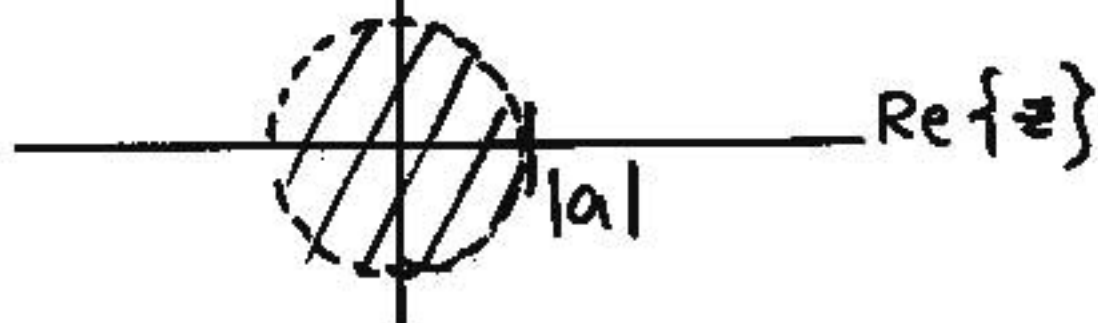
ROC:

$$|a^{-1}z| < 1$$

$$\left| \frac{z}{a} \right| < 1$$

$$|z| < |a|$$

ROC:  $|z| < |a|$   
 $\text{Im}\{z\}$



$X(z)$  is uniquely defined  
by functional form and ROC

$$3. \quad x[n] = a^n u[n] + b^n u[-n-1]$$

$\Rightarrow$  2-sided sequence

$$Z\{x[n]\} = X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$

$$\text{ROC: } \{|z| > |a|\} \cap \{|z| < |b|\}$$

• if  $|b| > |a|$ : ROC:  $|a| < |z| < |b|$

See pg. 158, Fig. 3.4 {annular region

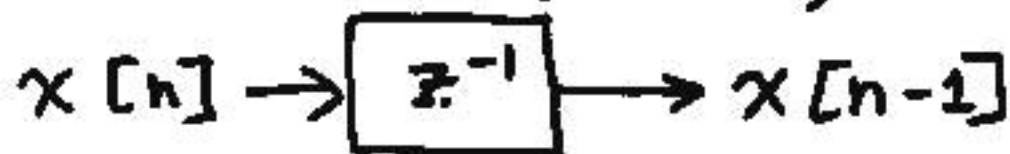
• if  $|b| < |a|$ ,  $\Rightarrow$  ROC =  $\emptyset$

## Properties of ZT:

- shifting property:

$$\mathcal{Z}\{x[n-k]\} = z^{-k} X(z)$$

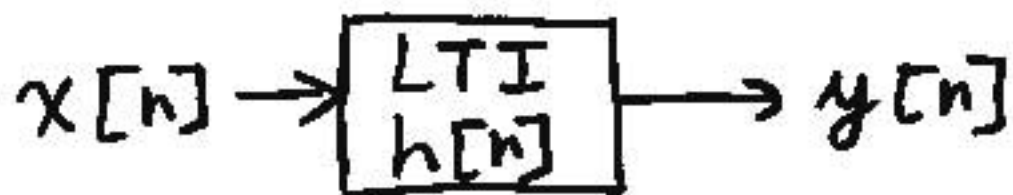
- for  $k=1$ :  $\mathcal{Z}\{x[n-1]\} = z^{-1} X(z)$



- Convolution Property:

$$\mathcal{Z}\{x[n] * h[n]\} = H(z) X(z)$$

See Table 3.2 on pg. 173 for further properties of ZT



$$Y(z) = H(z)X(z)$$

$$H(z) = Z\{h[n]\} = \frac{Y(z)}{X(z)}$$

- $z_0$  is a pole of  $H(z)$  if  $H(z_0) = \infty$
- $z_0$  is a zero of  $H(z)$  if  $H(z_0) = 0$

- ZT analysis of LTI System described by difference eqns

$$Z\{y[n]\} = Z\left\{-\sum_{k=1}^N a_k y[n-k]\right\} + Z\left\{\sum_{k=0}^M b_k x[n-k]\right\}$$

$$Y(z) = -\sum_{k=1}^N a_k z^{-k} Y(z) + \sum_{k=0}^M b_k z^{-k} X(z)$$

shift  
Prop.

• convolution property;

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$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

$$= b_0 \frac{z^{-M}}{z^{-N}} \cdot \frac{z^M + \left(\frac{b_1}{b_0}\right) z^{M-1} + \dots + \frac{b_M}{b_0}}{z^N + a_1 z^{N-1} + \dots + a_N}$$

$$= b_0 z^{N-M} \frac{(z - z_1)(z - z_2) \dots (z - z_M)}{(z - p_1)(z - p_2) \dots (z - p_N)}$$

- $z_k$ : zeros of  $H(z)$
- $p_k$ : poles of  $H(z)$



- assume poles are unique and  $M \leq N$  (if not, perform long division first)

(Keep in mind:  $f[n-k] \xleftrightarrow{z^{-k}} z^{-k}$ )

- partial fraction expansion

$$H(z) = A_1 \frac{z}{z-p_1} + A_2 \frac{z}{z-p_2} + \dots + A_N \frac{z}{z-p_N}$$

• where:  $A_k = \left. \frac{z-p_k}{z} H(z) \right|_{z=p_k}$

- Basic inversion result:

$$Z^{-1} \left\{ \frac{z}{z - p_k} \right\} = \begin{cases} p_k^n u[n], & \text{if } \text{ROC} \subset \{|z| > |p_k|\} \\ -p_k^n u[-n-1], & \text{if } \text{ROC} \subset \{|z| < |p_k|\} \end{cases}$$

- for repeated poles, see  
pp. 191-193 in P+H Text  
=> Example 3.4.7

- if  $z_i \neq p_j$   $i=1, \dots, M; j=1, \dots, N$   
then  $H(z) \Big|_{z=p_j} = \infty$
- ROC cannot contain a pole
- assume poles ordered as  
 $|p_1| \leq |p_2| \leq \dots \leq |p_N|$
- ROC must lie in an annular region:  
 $|p_k| < |z| < |p_{k+1}|$

- Causality requires  $h[n]=0$   
for  $n < 0$
- implies  $h[n]$  must be  
"right-sided" sequence
- ROC of  $H(z)$  must be  
 $|z| > |P_N|$
- $P_N$ : pole with largest magnitude

• Stability: requires ROC to include unit circle,  $|z|=1$

• Supporting argument:

• invoke triangle inequality  
( $|a+b| \leq |a| + |b|$ )

•  $|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n] z^{-n}|$

• on unit circle:  $|z^{-n}| = |z|^{-n} = \frac{1}{|z|^n} = 1$

• on unit circle:

$$|H(z)| \leq \underbrace{\sum_{n=-\infty}^{\infty} |h[n]|}_{\text{for BIBO Stability}} < \infty$$

• ROC must include unit circle  
for BIBO Stability



## Stability and Causality

- requires  $|z| > |p_N|$  must include unit circle,  $|z|=1$

$$\Rightarrow |p_N| < 1$$

$\Rightarrow$  all poles must be located within unit circle

- notes for distinct poles:

$$h[n] = \sum_{k=1}^N A_k p_k^n u[n]$$