EE538 DSP I

Module 3

Outline: Z-Transform

- Relevant P&M Sections:

  3.1, 3.2, 3.4.3, 3.6.1–3.6.4

Z-Transform

Recall:

$$e^{st} \longrightarrow \boxed{\begin{array}{c}\text{CT} \\ \text{LTI} \\ h_a(t)\end{array}} \longrightarrow H_a(s)e^{st}$$

$$H_a(s) = \underbrace{\mathcal{L}}_{\text{Laplace Transform}}\{h_a(t)\}$$

· Consider:

$$x[n]=a^n \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow y[n]=?$$

for all n

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

$$=\sum_{k=-\infty}^{\infty} h[k]\,a^{n-k}$$

$$=\left\{\sum_{k=-\infty}^{\infty} h[k]\,a^{-k}\right\} a^n$$

- Defining Z-Transform (ZT) of $h[n]$:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

$$\underset{-\infty < n < \infty}{x[n] = a^n} \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n]\end{matrix}} \longrightarrow y[n] = \underbrace{H(a)}_{\substack{\text{ZT of } h[n] \\ \text{evaluated at} \\ z=a}} a^n$$

- Examples.

"right-sided" sequence

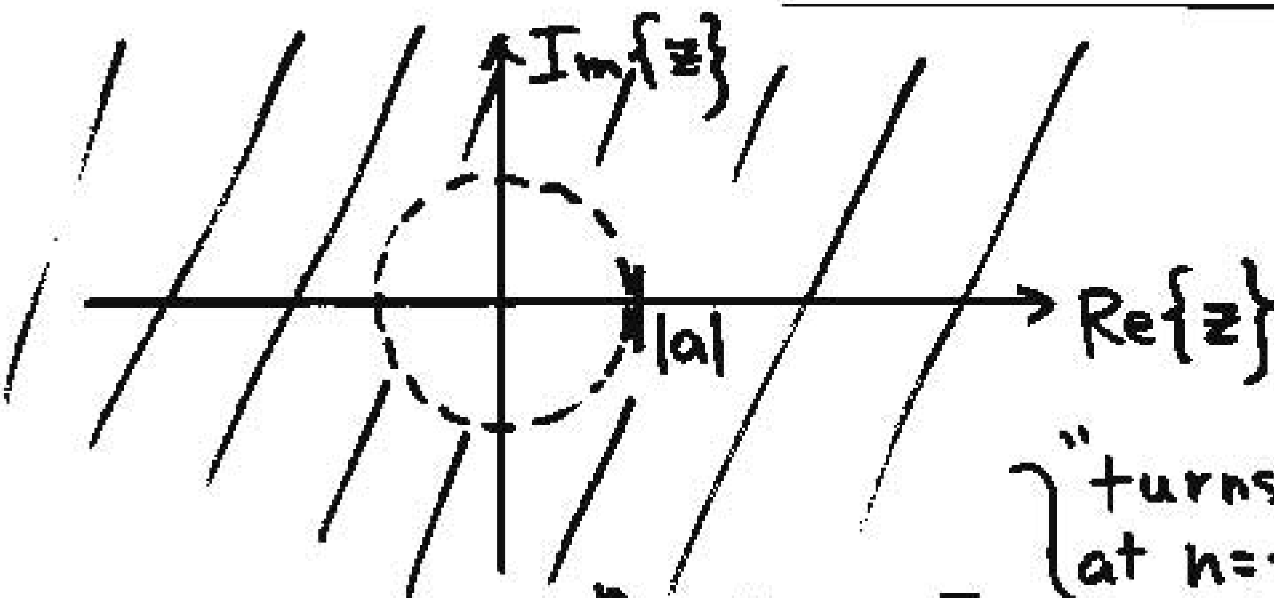
1. $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} \qquad |az^{-1}| < 1$$

- Region of Convergence (ROC):
- any value of z for which $X(z) < \infty$

$$\left|\frac{a}{z}\right| < 1 \Rightarrow |a| < |z| \Rightarrow |z| > |a|$$



2. $x[n] = -a^n u[-n-1]$

"turns on" at $n=-\infty$ and "shuts off" at $n=0$

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

change of variables:
$n' = -n$
$(n = -n')$

$$X(z) = -\sum_{n'=1}^{\infty} (az^{-1})^{-n'}$$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^{n}$$

$$= \frac{-1}{1 - a^{-1}z} + 1$$

$$= \frac{-a}{a - z} + 1$$

$$= \frac{a}{z-a} + \frac{z-a}{z-a} = \frac{z}{z-a}$$

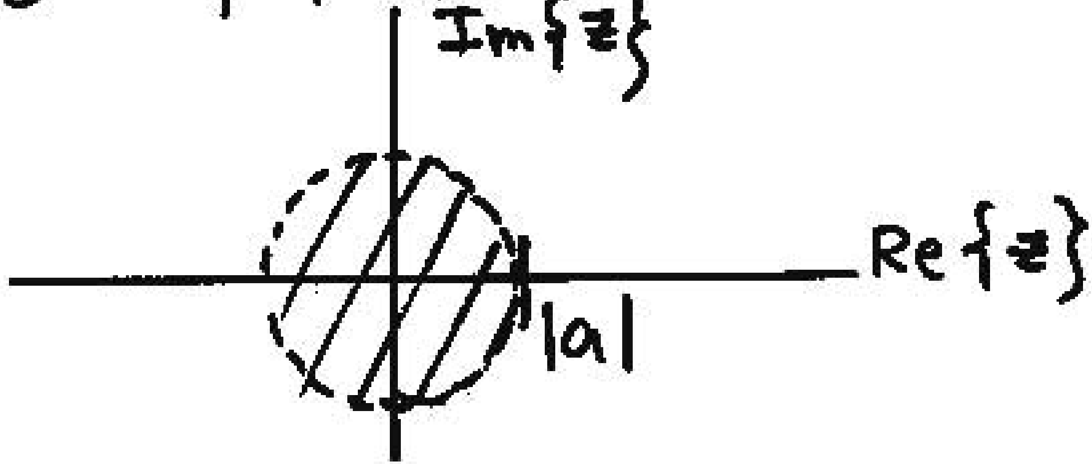
ROC:

$$|a^{-1}z| < 1$$

$$\left|\frac{z}{a}\right| < 1$$

$$|z| < |a|$$

ROC: $|z| < |a|$



X(z) is uniquely defined by functional form and ROC

3. $x[n] = a^n u[n] + b^n u[-n-1]$

$\Rightarrow$ 2-sided sequence

$$\mathcal{Z}\{x[n]\} = X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$

ROC: $\{|z| > |a|\} \cap \{|z| < |b|\}$

- if $|b| > |a|$: ROC: $|a| < |z| < |b|$

See pg. 158, Fig. 3.4 {annular region

- if $|b| < |a|$, $\Rightarrow$ ROC $= \emptyset$

Properties of ZT:

- shifting property:

$$\mathcal{Z}\{x[n-k]\} = z^{-k}X(z)$$

- for $k=1$: $\mathcal{Z}\{x[n-1]\} = z^{-1}X(z)$

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-1]$$

- Convolution Property:

$$\mathcal{Z}\{x[n] * h[n]\} = H(z)X(z)$$

See Table 3.2 on pg. 173 for further properties of ZT

$$x[n] \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h[n]\end{array}} \longrightarrow y[n]$$

$$Y(z) = H(z)\,X(z)$$

$$H(z) = \mathcal{Z}\{h[n]\} = \frac{Y(z)}{X(z)}$$

- z_0 is a pole of $H(z)$ if $H(z_0) = \infty$
- z_0 is a zero of $H(z)$ if $H(z_0) = 0$

• ZT analysis of LTI System described by difference eqns

$$\mathcal{Z}\{y[n]\} = \mathcal{Z}\left\{-\sum_{k=1}^{N} a_k\, y[n-k]\right\} + \mathcal{Z}\left\{\sum_{k=0}^{M} b_k\, x[n-k]\right\}$$

$$Y(z) = -\sum_{k=1}^{N} a_k\, z^{-k}\, Y(z) + \sum_{k=0}^{M} b_k\, z^{-k}\, X(z)$$

shift prop.

- Convolution property:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= b_0 \frac{z^{-M}}{z^{-N}} \cdot \frac{z^M + \left(\frac{b_1}{b_0}\right) z^{M-1} + \dots + \frac{b_M}{b_0}}{z^N + a_1 z^{N-1} + \dots + a_N}$$

$$= b_0 z^{N-M} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

- z_k: zeroes of $H(z)$
- p_k: poles of $H(z)$

- assume poles are unique and $M \leq N$ (if not, perform long division first)

(Keep in mind: $\delta[n-k] \overset{ZT}{\longleftrightarrow} z^{-k}$)

- partial fraction expansion

$$H(z) = A_1 \frac{z}{z-p_1} + A_2 \frac{z}{z-p_2} + \dots + A_N \frac{z}{z-p_N}$$

- where: $A_k = \left. \dfrac{z-p_k}{z} H(z) \right|_{z=p_k}$

- Basic inversion result:

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-p_k}\right\} = \begin{cases} p_k^n\, u[n], & \text{if ROC} \subset \{|z|>|p_k|\} \\ -p_k^n\, u[-n-1], & \text{if ROC} \subset \{|z|<|p_k|\} \end{cases}$$

- for repeated poles, see pp. 191-193 in P+M Text
  $\Rightarrow$ Example 3.4.7

- if $z_i \neq p_j \quad i=1,\dots,M; \ j=1,\dots,N$
  then $H(z)\big|_{z=p_j} = \infty$
- ROC cannot contain a pole
- assume poles ordered as
  $|p_1| \le |p_2| \le \dots \le |p_N|$
- ROC must lie in an annular region: $|p_k| < |z| < |p_{k+1}|$

- <u>Causality</u> requires $h[n]=0$ for $n<0$
- implies $h[n]$ must be "right-sided" sequence
- ROC of $H(z)$ must be
  $|z| > |p_N|$
- p_N: pole with largest magnitude

- **Stability**: requires ROC to include unit circle, $|z| = 1$
- Supporting argument:
- invoke triangle inequality ($|a+b| \le |a| + |b|$)
- $|H(z)| \le \sum_{n=-\infty}^{\infty} |h[n] z^{-n}|$
- on unit circle: $|z^{-n}| = |z|^{-n} = 1^{n} = 1$

- on unit circle:

$$|H(z)| \le \underbrace{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}_{\text{for BIBO Stability}}$$

- ROC must include unit circle for BIBO Stability

• Stability and Causality

• requires $|z| > |p_N|$ must include unit circle, $|z| = 1$

$\Rightarrow |p_N| < 1$

$\Rightarrow$ all poles must be located within unit circle

• note: for distinct poles:

$$h[n] = \sum_{k=1}^{N} A_k \, p_k^n \, u[n]$$