EE538 DSP I

Module 28

Outline

- Relationship between AR spectral estimation and linear Prediction — Sect. 11.2.1
- Levinson-Durbin Algorithm for solving Yule-Walker Eqns. — Sect. 11.3.1

- Relationship between AR spect. est. and LP
- consider predicting $x[n]$ in terms of m past samples

$$\hat{x}[n] = -\sum_{k=1}^{m} a_m(k)\, x[n-k]$$

- choose m-th order prediction coefficients to minimize MSE

$$\mathcal{E} = E\{|e[n]|^2\}$$

$$= E\{|x[n]-\hat{x}[n]|^2\}$$

- for sake of simplicity, assume $x[n]$ is real-valued

$$\mathcal{E} = E\left\{\left(x[n] + \sum_{k=1}^{m} a_m(k)\, x[n-k]\right)^2\right\}$$

$$\frac{\partial \mathcal{E}}{\partial a_m(\ell)} = E\left\{\frac{\partial}{\partial a_m(\ell)}\left[x[n] + \sum_{k=1}^{m} a_m(k)\, x[n-k]\right]^2\right\}$$

$$= E\left\{2\underbrace{\left[x[n] + \sum_{k=1}^{m} a_m(k)\, x[n-k]\right]}_{e[n]} x[n-\ell]\right\} = 0$$

$$r_{xx}[\ell] = -\sum_{k=1}^{m} a_m(k)\, r_{xx}[\ell-k]$$

$$\ell = 1, 2, \ldots, m$$

- m eqns. in m unknowns

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[-1] & \cdots & r_{xx}[1-m] \\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[2-m] \\ \vdots & \vdots & & \vdots \\ r_{xx}[m-1] & r_{xx}[m-2] & \cdots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_m(1) \\ a_m(2) \\ \vdots \\ a_m(m) \end{bmatrix} = - \begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \\ \vdots \\ r_{xx}[m] \end{bmatrix}$$

Symmetric-Toeplitz $\left\{ \underline{R}_m \underline{a}_m = -\underline{r}_m \right.$

- if $m=p$ and $x[n]$ is an AR(p) process, same eqns. as that relating $r_{xx}[l]$, $l=0,1,\ldots,p$ to a_k, $k=1,\ldots,p$ $\Rightarrow a_p(k) = a_k$, $k=1,2,\ldots,p$

- minimum prediction error:

$$\mathcal{E}_m^{min} = E\left\{ \left[x[n] + \sum_{k=1}^{m} a_m x[n-k]\right] \cdot \left[x^*[n] + \sum_{k=1}^{m} a_m^*(k) x^*[n-k]\right] \right\}$$

$$\mathcal{E}_m^{min} = E\left\{\left[x[n] + \sum_{k=1}^{m} a_m(k)\, x[n-k]\right] x^*[n]\right\}$$

$$+ E\left\{\left[x[n] + \sum_{k=1}^{m} a_m(k)\, x[n-k]\right] \sum_{k=1}^{m} a_m^*(k)\, x^*[n-k]\right\}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\sum_{k=1}^{m} a_m^*(k)\, \underbrace{E\{e[n]\, x^*[n-k]\}}_{=0 \text{ orthogonality principle}}}$$

$$\mathcal{E}_m^{min} = r_{xx}[0] + \sum_{k=1}^{m} a_m(k)\, \underbrace{r_{xx}^*(k)}_{r_{xx}(-k)}$$

- $\underline{R}_m \underline{a}_m = -\underline{r}_m$
- because: i) $\underline{R}_m$ is Toeplitz-Hermitian

  ii.) $\underline{r}_m$ and 1^{st} col. of $\underline{R}_m$ have all but one element in common
- this set of eqns. may be solved efficiently via Levinson-Durbin algorithm — see Sect. 11.3.1 for derivation

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- L-D algorithm sequentially solves Y-W eqns. for progressively higher predictor orders

$$r_{xx}[0]\, a_1(1) = -r_{xx}[1]$$ 1st-order predictor

$$\begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_2(1) \\ a_2(2) \end{bmatrix} = -\begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \end{bmatrix}$$

$$\begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] & r_{xx}^*[2] \\ r_{xx}[1] & r_{xx}[0] & r_{xx}^*[1] \\ r_{xx}[2] & r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_3(1) \\ a_3(2) \\ a_3(3) \end{bmatrix} = -\begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \\ r_{xx}[3] \end{bmatrix}$$

Outline of L-D Algorithm

- Initialization: $a_0(0) = 0$ $\quad$ 0^{th}-order predictor

$$\mathcal{E}_0^{min} = E\{[x[n]-0]^2\} = r_{xx}[0]$$

- for $m = 1, \ldots, P$:

$$a_m(m) = \frac{-\left\{r_{xx}[m] + \sum_{k=1}^{m-1} a_{m-1}(k) r_{xx}[m-k]\right\}}{\mathcal{E}_{m-1}^{min}}$$

- for $k = 1, \ldots, m-1$:

$$a_m(k) = a_{m-1}(k) + a_m(m) a_{m-1}^*(m-k)$$

- end

$$\mathcal{E}_m^{min} = \mathcal{E}_{m-1}^{min}\{1 - |a_m(m)|^2\}$$

- end

Numerical Example:

Given: $r_{xx}[0] = 1$; $r_{xx}[1] = \frac{1}{2}$; $r_{xx}[2] = \frac{1}{8}$ for an AR(2) process.

1. Determine AR model parameters a_1 and a_2, and σ_w^2 = power of i.i.d. noise input to the 2-pole filter that generated $x[n]$, an AR(2) process

2. Determine $r_{xx}[3]$.

3. Closed-form expression for

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-j\omega m}$$

$$a_1(1) = \frac{-r_{xx}[1]}{\varepsilon_0^{min}} = \frac{-r_{xx}[1]}{r_{xx}[0]} = \frac{-\frac{1}{2}}{1} = -\frac{1}{2}$$

- minimum prediction error:

$$\varepsilon_1^{min} = \varepsilon_0^{min}\left\{1-|a_1(1)|^2\right\} = r_{xx}[0]\left\{1-\left(-\frac{1}{2}\right)^2\right\}$$

$$= 1\left\{\frac{3}{4}\right\} = \frac{3}{4}$$

- 2^{nd}-order predictor:

$$a_2(2) = \frac{-\left\{r_{xx}[2] + a_1(1)\, r_{xx}[1]\right\}}{\varepsilon_1^{min}}$$

$$a_2(2) = \frac{-\left\{\frac{1}{8} - \frac{1}{2}\left(\frac{1}{2}\right)\right\}}{3/4} =$$

$$= -\frac{4}{3}\left\{\frac{1}{8} - \frac{2}{8}\right\} = \frac{1}{6} = a_2$$

$$a_2(1) = a_1(1) + a_2(2)\, a_1^*(1)$$

AR model parameters

$$= a_1(1)\left\{1 + a_2(2)\right\} =$$

$$= \left(-\frac{1}{2}\right)\left\{1 + \frac{1}{6}\right\} = \frac{-7}{12} = a_1$$

$$\sigma_2^{min} = \sigma_1^{min}\left\{1 - |a_2(2)|^2\right\}$$

$$\varepsilon_2^{min} = \frac{3}{4}\left\{1-\left(\frac{1}{6}\right)^2\right\} = \frac{3(35)}{(4)36} = \frac{35}{48}$$

$$= \sigma_w^2$$

• check:

$$\sigma_w^2 = r_{xx}[0] + \sum_{k=1}^{2} a_k r_{xx}^*[k]$$

$$= 1 + \left(-\frac{7}{12}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{8}\right)$$

$$= \frac{48 - 14 + 1}{48} = \frac{35}{48} \checkmark$$

$$\left(a_1 = a_2(1) \text{ and } a_2 = a_2(2)\right)$$

check:

$$\begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_2(1) \\ a_2(2) \end{bmatrix} = - \begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{7}{12} \\ \frac{1}{6} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{8} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

2. $r_{xx}[3] = ? \quad -\sum_{k=1}^{p} a_k r_{xx}[m-k]$

$$r_{xx}[3] = -a_1 r_{xx}[2] - a_2 r_{xx}[1]$$

$$r_{xx}[m] = -\sum_{k=1}^{p} a_k r_{xx}[m-k] \quad \Big\} \text{ for } m>0$$

$$r_{xx}[3] = \frac{7}{12}\left(\frac{1}{8}\right) - \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)$$

$$= \frac{7-8}{96} = -\frac{1}{96}$$

3. $$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-jm\omega}$$

$$= \frac{\sigma_w^2}{\left|1 + \sum_{k=1}^{2} a_k e^{-jk\omega}\right|^2}$$

$$= \frac{35/48}{\left|1 - \frac{7}{12} e^{-j\omega} + \frac{1}{6} e^{-j2\omega}\right|^2}$$