EE538 DSP I

Module 25

Outline

- Extrapolation of the autocorrelation for a SoS signal
- Basics of Autoregressive (AR) spectral estimation

Test 3 - Session 26

Same format as previous exams., e.g. open book

Problems Topics:

1. Radix-2 FFT
2. DFT and time-domain aliasing
3. Windows and symmetric linear-phase FIR Filter

- problem with nonparametric spectral estimators: inherently assume

$$r_{xx}[m] = 0 \quad \text{for } m > M$$

where: $M \leq N-1$

- in fact, though, for SoS (w/o noise)

$$r_{xx}[m] = \sum_{i=1}^{p} P_i e^{j\omega_i m}$$

- does not decay to zero as $m \to \infty$

- in general, sharp spectral peaks in $S_{xx}(\omega)$ imply $r_{xx}[m]$ takes a "long time" to decay to zero as $m \to \infty$
- interesting observation re: $r_{xx}[m]$ for SoS model ⇒ it satisfies recursive relationship

$$r_{xx}[m] = -\sum_{k=1}^{P} a_k \, r_{xx}[m-k] \quad \text{for } m \geq P$$

• where: a_k, $k=1,\dots,P$ are the coefficients of the polynomial

$$(z-e^{j\omega_1})(z-e^{j\omega_2})\cdots(z-e^{j\omega_P})$$

$$= z^P + a_1 z^{P-1} + a_2 z^{P-2} + \dots + a_{P-1} z + a_P$$

$$1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_{P-1} z^{-(P-1)} + a_P z^{-P} = 0$$

for $z_i = e^{j\omega_i}$

$$\left\{ 1 + a_1 e^{j\omega_i(-1)} + a_2 e^{j\omega_i(-2)} + \dots + a_{p-1} e^{j\omega_i(-(p-1))} + a_p e^{j\omega_i(-p)} \right\} = 0$$

- multiply both sides by $P_i e^{j\omega_i m}$
- sum over all i (since $\sum 0 = 0$)

$$\sum_{i=1}^{p} P_i e^{j\omega_i m} + a_1 \sum_{i=1}^{p} P_i e^{j\omega_i (m-1)} + \dots + a_{p-1} \sum_{i=1}^{p} P_i e^{j\omega_i (m-(p-1))} + a_p \sum_{i=1}^{p} P_i e^{j\omega_i (m-p)} = 0$$

- ultimately:

$$r_{xx}[m] = -\sum_{k=1}^{P} a_k \, r_{xx}[m-k] \quad \text{for } m \geq P$$

- End of Proof

- Thus: if:
- if we estimate $r_{xx}[m]$ for $m = 0, 1, \ldots, P$ AND
- determine the coefficients a_k, $k = 1, \ldots, P$

- we could extrapolate $r_{xx}[m]$ for all m, thereby averting a windowing effect
- ostensibly use recursion to determine $r_{xx}[m]$ for $m \geq P$ and form $S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-j\omega m}$
- however, in case of SoS, typically interested in frequencies, ω_i, which may be determined from roots of $z^p + a_1 z^{p-1} + \ldots + a_{p-1} z + a_p$

- note: $r_{xx}[-m] = r_{xx}^*[m]$
- How do we estimate $a_k,\ k = 1, \ldots, P$ given $r_{xx}[m],\ m = 0, 1, \ldots, P$
- $r_{xx}[m] = -\sum_{k=1}^{P} a_k\, r_{xx}[m-k]$
- write out P eqns. in P unknowns $\Rightarrow m = 1, 2, \ldots, P$

$$
\underbrace{\begin{bmatrix} r_{xx}[0] & r_{xx}[-1] & \cdots & r_{xx}[-(p-1)] \\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[-(p-2)] \\ \vdots & \vdots & \cdots & \vdots \\ r_{xx}[p-1] & r_{xx}[p-2] & \cdots & r_{xx}[0] \end{bmatrix}}_{\underline{R}} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} \hat{=} - \begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \\ \vdots \\ r_{xx}[p] \end{bmatrix}
$$

$$
\underset{p \times p}{\underline{R}} \; \underset{p \times 1}{\underline{a}} = - \underset{p \times 1}{\underline{r}}
$$

See SoSextrap.m at web site

- these observations for the SoS model are the basis for the following spectral estimation techniques:

- Auto Regressive (AR) Method
- MUSIC (Multiple SIgnal Classification) - Sect. 12.5.3
- ESPRIT (Estimation of Signal Parameters by Rotational Invariance Techniques) Sect. 12.5.4

- AR Spectral Estimation

- for sole purpose of matching the spectral density of a WSS random process, $x[n]$, we model $x[n]$ as having been generated as:

$$v[n] \longrightarrow \boxed{\begin{array}{c}\text{All-pole}\\ \text{IIR Filter}\end{array}} \longrightarrow x[n]$$

where: $r_{vv}[m] = \sigma_w^2 \delta[m]$

AR(P) process

- $$x[n] = -\sum_{k=1}^{p} a_k \, x[n-k] + v[n]$$
- can show that $r_{xx}[m]$ satisfies

$$r_{xx}[m] = -\sum_{k=1}^{p} a_k \, r_{xx}[m-k] + \sigma_w^2 \delta[m]$$

- in freq. domain:

$$X(\omega) = H(\omega)\, V(\omega)$$

- where: $H(\omega) = \dfrac{1}{1+\sum_{k=1}^{p} a_k e^{-jk\omega}}$

- from basis random process theory

$$S_{xx}(\omega) = |H(\omega)|^2 S_{vv}(\omega)$$

- where: $S_{\nu\nu}(\omega) = \text{DTFT}\{r_{\nu\nu}[m]\}$

$$= \text{DTFT}\{\sigma_w^2 \delta[m]\}$$

$$= \sigma_w^2 \text{ for all } \omega$$

- thus:

$$S_{xx}(\omega) = \text{DTFT}\{r_{xx}[m]\}$$

$$= \frac{\sigma_w^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-jk\omega}\right|^2}$$