EE538 DSPI
Module 24

Outline:

- Nonparametric estimation of autocorrelation and power spectrum
  - Sect. 12.1, 12.2.3
- Sum of sinewaves model – Sect. 12.5.2
- DT random processes – Appendix A

- Parametric Spectral Estimation
- with emphasis on harmonic retrieval methods
- i.e., estimating frequencies and amplitudes of superimposed sinewares (or geometric sequences) embedded in noise

- parametric = model-based
- Why parametric?
- to achieve higher resolution of sharp spectral peaks than that achieved with nonparametric spectral estimation
- the latter is primarily Wiener-Kinchen Theorem

- power or energy density spectrum of a DT signal $x[n]$ is

$$S_{xx}(\omega) = |X(\omega)|^2$$

- where: $x[n] \xleftrightarrow{DTFT} X(\omega)$

- $$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-jm\omega}$$

- where:

$$r_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n-m]$$

- Stochastic Case:

$$S_{xx}(\omega) = E\{|X(\omega)|^2\}$$

- $r_{xx}[m] \xleftrightarrow{DTFT} S_{xx}(\omega)$

- where:

$$r_{xx}[m] = E\{x[n]\,x^*[n-m]\}$$

- where: $E\{\cdot\}$ = expectation operator – see Appendix A
- $x[n]$ assumed to wide-sense stationary

- in practical case, given N samples of a wide sense stationary random process $x[n]$, $n = 0, 1, \ldots, N-1$
- autocorrelation is estimated as

$$\hat{r}_{xx}[m] = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x[n] \, x^*[n-m]$$

for $m = 0, 1, \ldots, N-1$

- note: upper limit in sum
- for a given m, there are N−m pairs of samples separated by m units of DT to average over in order to:
- achieve uncorrelatedness/independence amongst different constituent signals comprising x[n]
- average out the effects of noise

Case Study : Sum of Sinewaves (SoS) in Noise

$$x[n] = \sum_{i=1}^{P} A_i e^{j(\omega_i n + \phi_i)} + \nu[n]$$

- A_i, ω_i : deterministic but unknown
- ϕ_i : independent and identically (i.i.d.) uniform random variables over $[0, 2\pi)$

$\nu[n]$: contribution from noise

- assumptions :

i.) zero mean : $E\{\nu[n]\} = 0$

ii.) uncorrelated / independent with signal

$$E\{\nu[n]\, s^*[n-m]\} = 0 \quad \text{for all } m$$

- where:

$$s[n] = \sum_{i=1}^{P} A_i e^{j\phi_i} e^{j\omega_i n}$$

iii.) $\nu[n]$ is a DT WSS random process

$$r_{\nu\nu}[m] = E\{\nu[n]\nu^*[n-m]\}$$

- typically assume white noise:

$$r_{\nu\nu}[m] = \sigma_w^2\,\delta[m]$$

- $E\{\cdot\}$ is a linear operator:

$$E\left\{\sum_{i=1}^{Q} a_i X_i\right\} = \sum_{i=1}^{Q} a_i\, E\{X_i\}$$

- "true" autocorrelation for SoS in noise:

$$r_{xx}[m] = E\{x[n]\,x^*[n-m]\}$$

$$= E\left\{\sum_{i=1}^{P} A_i e^{j\phi_i} e^{j\omega_i n} \sum_{\ell=1}^{P} A_\ell e^{-j\phi_\ell} e^{-j\omega_\ell (n-m)}\right\}$$

$$= \sum_{i=1}^{P} \sum_{\ell=1}^{P} A_i A_\ell\, e^{j\omega_i n} e^{-j\omega_\ell (n-m)}$$

$$\cdot \underbrace{E\{e^{j\phi_i} e^{-j\phi_\ell}\}}_{\delta[i-\ell]} +$$

$$+ E\{v[n]s^*[n-m]\} + E\{s[n]v^*[n-m]\}$$
$$+ E\{v[n]v^*[n-m]\}$$

- due to i.i.d. assumption, for $i \neq l$:

$$E\{e^{j\phi_i} e^{-j\phi_l}\} = E\{e^{j\phi_i}\} E\{e^{-j\phi_l}\}$$

$$E\{e^{j\phi_i}\} = \int_0^{2\pi} \frac{1}{2\pi} e^{j\phi} d\phi$$

$$= \frac{1}{j} \frac{1}{2\pi} e^{j\phi}\Big|_0^{2\pi} = \frac{1}{2\pi j}\{1-1\} = 0$$

- thus: $\quad P_i = A_i^2, \; i = 1, \ldots, P$

$$r_{xx}[m] = \sum_{i=1}^{P} P_i \, e^{j\omega_i m} + \sigma_w^2 \delta[m]$$

for all m

- Compare to time-averaged estimate of autocorrelation
- substitute SoS in noise model into

$$\hat{r}_{xx}[m] = \sum_{n=0}^{N-m-1} x[n] \, x^*[n-m] \, \frac{1}{N-m}$$

$$\hat{r}[m] =$$

$$\sum_{i=1}^{P} \sum_{l=1}^{P} A_i A_l e^{j(\phi_i - \phi_l)} e^{j\omega_l m} \cdot \frac{1}{N-m} \sum_{n=0}^{N-m-1} e^{j(\omega_i - \omega_l)n}$$

$$+ \frac{1}{N-m} \sum_{n=0}^{N-m-1} s[n] \nu^*[n-m] +$$

$$\frac{1}{N-m} \sum_{n=0}^{N-m-1} \nu[n] s^*[n-m] + \frac{1}{N-m} \sum_{n=0}^{N-m-1} \nu[n] \nu^*[n-m]$$

- for lag value m, need to avg. over enough DT sample pairs separated by m so that:

$$\frac{1}{N-m}\sum_{n=0}^{N-m-1} e^{j(\omega_i-\omega_\ell)n}$$

$$= \frac{1}{N-m}\cdot\frac{1-e^{j(\omega_i-\omega_\ell)(N-m)}}{1-e^{j(\omega_i-\omega_\ell)}}$$

$$= \frac{1}{N-m}\,\frac{\sin\left[\frac{N-m}{2}(\omega_i-\omega_\ell)\right]}{\sin\left[\frac{1}{2}(\omega_i-\omega_\ell)\right]}$$

$$\lim_{N\to\infty} \quad = \quad \delta[i-\ell]$$

for fixed $m<\infty$

$$= \frac{1}{\sin\left[\frac{1}{2}(\omega_i-\omega_\ell)\right]}\cdot\frac{\sin\left[\frac{N-m}{2}(\omega_i-\omega_\ell)\right]}{N-m}$$

- as lag value m increases, less number of pairs to average over
- estimates of $r_{xx}[m]$ become increasingly unreliable as m increases

- general form of nonparametric spectral estimator:

$$\hat{S}_{xx}(\omega) = \sum_{m=-(N-1)}^{N-1} \hat{r}_{xx}[m]\, w[m]\, e^{-j\omega m}$$

- where $w[m]$ is a real-valued, symmetric $(w[-m] = w[m])$, tapered $(w[m+1] \leq w[m])$, and $W(\omega) = \mathrm{DTFT}\{w[m]\} > 0$ for all ω

- e.g. Bartlett (Triangle) Window

- $w[m] = \dfrac{N-|m|}{N}$, $m = 0, 1, \ldots, M$

  where: $M \leq N-1$

  $= 0$ otherwise

- Expected value of nonparametric spectral estimator (See Sect. 12.2.3)

- $$E\{\hat{S}_{xx}(f)\} = \sum_{m=-(N-1)}^{N-1} E\{\hat{r}_{xx}[m]\} w[m] e^{-j\omega m}$$

$$= \sum_{m=-\infty}^{\infty} r_{xx}[m]\, w[m]\, e^{-j\omega m}$$

$$= S_{xx}(\omega) \circledast W(\omega)$$

true spectrum

periodic convolution

see pg. 303 of P&M text

- convolution operation with $W(\omega)$ causes "smearing" of fine spectral details <u>and</u> "ringing" - noisy background that can mask weak signal components

- Return to SOS in noise Case Study

$$S_{xx}(f) = \text{DTFT}\{r_{xx}[m]\}$$

$$= \sum_{i=1}^{p} P_i \, 2\pi\delta_a(\omega-\omega_i) + \sigma_w^2$$

for $|\omega| < \pi$

Dirac Delta function

$$E\{\hat{S}_{xx}(\omega)\} = \sum_{i=1}^{p} P_i \, W(\omega-\omega_i) + \sigma_w^2 w[0]$$

• Example: $P = 3$