EE538 DSP I

Module 22

Outline:

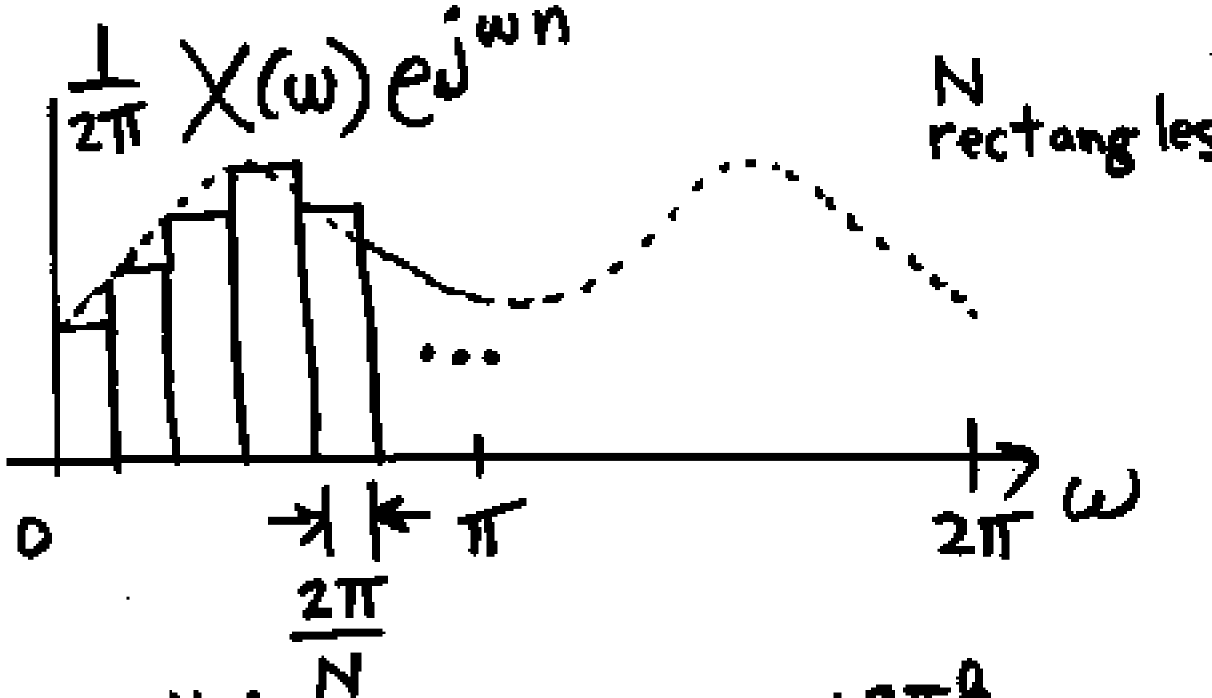
- Inverse DFT and time-domain aliasing – Sect. 5.1.2, 5.2.2, 5.3.1
- Analysis of Windowing and Truncation Effects – Sect. 8.2.2

- Inverse DFT
- recall inverse DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

- consider discretizing the integral a la Riemann sum



$$x_p[n] = \sum_{k=0}^{N-1} \underbrace{\frac{1}{2\pi} X\left(\frac{2\pi k}{N}\right) e^{j\frac{2\pi k}{N} n}}_{\text{height}} \underbrace{\frac{2\pi}{N}}_{\text{width}}$$

$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_N(k)\, e^{j\frac{2\pi k}{N} n}$$

- **Question**: how does $x_p[n]$ relate to $x[n]$?

- **Answer**: $x_p[n]$ is periodic extension of $x[n]$

$$x_p[n] = \sum_{\ell=-\infty}^{\infty} x[n+\ell N]$$

Proof: Math preamble

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}n} = \frac{1}{N}\,\frac{1-e^{j2\pi n}}{1-e^{j\frac{2\pi}{N}n}}$$

$$= \begin{cases} 1, & n = lN,\ l\text{-integer} \\ 0, & \text{otherwise} \end{cases}$$

$$= \sum_{l=-\infty}^{\infty} \delta[n+lN]$$

$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_N(k)\, e^{j\frac{2\pi k n}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left\{\sum_{n'=0}^{L-1} x[n']\, e^{-j\frac{2\pi k}{N}n'}\right\} e^{j\frac{2\pi k}{N}n}$$

$$= \sum_{n'=0}^{L-1} x[n']\, \frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-n')}$$

$$= \sum_{n'=0}^{L-1} x[n'] \sum_{\ell=-\infty}^{\infty} \delta[n-n'+\ell N]$$

$$x_p[n] = \sum_{\ell=-\infty}^{\infty} \left\{ \sum_{n'=0}^{L-1} x[n']\,\delta[n-n'+\ell N] \right\}$$

$$= \sum_{\ell=-\infty}^{\infty} x[n] * \delta[n+\ell N]$$

$$= \sum_{\ell=-\infty}^{\infty} x[n+\ell N]$$

- End of Proof

Basic DFT Result:

$$\sum_{\ell=-\infty}^{\infty} x[n-\ell N]\, w_R[n] \xleftrightarrow[N]{\text{DFT}} X(\omega)\Big|_{\omega = k\frac{2\pi}{N}} \triangleq X_N(k)$$

$$w_R[n] = u[n] - u[n-N] \qquad k = 0, 1, \ldots, N-1$$

where: $x[n] \xleftrightarrow{\text{DTFT}} X(\omega)$

If $N > L$ = "length" of signal, then:

$$X(\omega) = \sum_{k=0}^{N-1} \underbrace{X_N(k)}_{X\left(\frac{2\pi k}{N}\right)} \frac{\sin\left(\frac{N}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)}{N\sin\left(\frac{1}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)} e^{-j\frac{(N-1)}{2}\left(\omega - k\frac{2\pi}{N}\right)}$$

and no time-domain aliasing

- If $x[n]$ is of length $L < N$, then

$$x_p[n] = \begin{cases} x[n], & n = 0, 1, \ldots, L-1 \\ 0, & n = L, \ldots, N-1 \end{cases}$$

- note: $x_p[n]$ is periodic with period N
- in this case, have exact transform pair - i.e., if sample $X(\omega)$ at $N > L$ equi-spaced points over $0 < \omega < 2\pi$:

- N-pt. DFT:

$$X_N(k) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N} n}$$

$k = 0, 1, \ldots, N-1$

- N-pt. Inverse DFT:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_N(k)\, e^{j\frac{2\pi k}{N} n}$$

$n = 0, 1, \ldots, N-1$

- <u>exact</u> transform pair!!

- Use of DFT to compute linear convolution
- Recall:

$$y[n] = x[n] * h[n] \xleftrightarrow{\text{DTFT}} Y(\omega) = X(\omega)H(\omega)$$

- Consider:

$$x[n] \xleftrightarrow[N]{\text{DFT}} X_N(k)$$

$$h[n] \xleftrightarrow[N]{\text{DFT}} H_N(k)$$

$$y_p[n] = \sum_{l=-\infty}^{\infty} y[n+lN] \xleftrightarrow[N]{\text{DFT}} X_N(k)\,H_N(k)$$

- assume:
- $x[n]$ is of length L ($N-L$ zeroes padded)
- $h[n]$ is of length M ($N-M$ zeroes padded)
- $y[n] = x[n] * h[n]$ is of length $M+L-1$
- recall:

$$y_p[n] = \sum_{\ell=-\infty}^{\infty} y[n+\ell N]$$

- as long as $N \geq M+L-1$, then

$$y_p[n] = \begin{cases} y[n], & n=0,1,\dots,M+L-2, \\ 0, & n=M+L-1,\dots,N-1 \end{cases}$$

- See Examples 5.3.1 and 5.3.2 in Text on pp. 427-430
- See timealias.m at web site
- See Problem 1 on Exam 3 of Fall 1994

- if $\max\{L, M\} \le N \le M + L - 1$,

then:

$$y_p[n] = y[n] + y[n+N]$$

for $n = 0, 1, \ldots, N-1$

- time-domain aliasing!