EE538 DSP I

Module 2

Outline

- Correlation of DT Signals
  - Sect. 2.6 of P+M Text
  - Psuedo-noise sequences
  - application to radar
  - application to spread spectrum communications - see cdmaeg.m

- Define cross-correlation:

$$r_{xy}(\ell) = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n-\ell]$$

Deterministic Case

- auto-correlation:

$$r_{xx}(\ell) = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n-\ell]$$

- for each lag ℓ:
  - shift to right by ℓ
  - point-wise multiply
  - sum

- relative to convolution, missing initial fold step

$$r_{xy}(\ell) = x(\ell) * y^*(-\ell)$$

$$r_{xx}(\ell) = x(\ell) * x^*(-\ell)$$

- <u>Example</u>: Psuedo-Noise (PN) Sequence

$$x[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, -1, -1, 1, -1\}$$

$$\{1, 1, 1, -1, -1, 1, -1\}$$

$$\ell = 0: \{1, 1, 1, -1, -1, 1, -1\}$$

$\ell = 0$:

$r_{xx}[0] = 7$

$$\ell = 1: \{1, 1, 1, -1, -1, 1, -1\}$$

$\ell = 1$:

$r_{xx}[1] = 0$

$$\ell = 2: \{1, 1, 1, -1, -1, 1, -1\}$$

$r_{xx}[2] = -1$

$r_{xx}[3] = 0$

$r_{xx}[4] = -1$

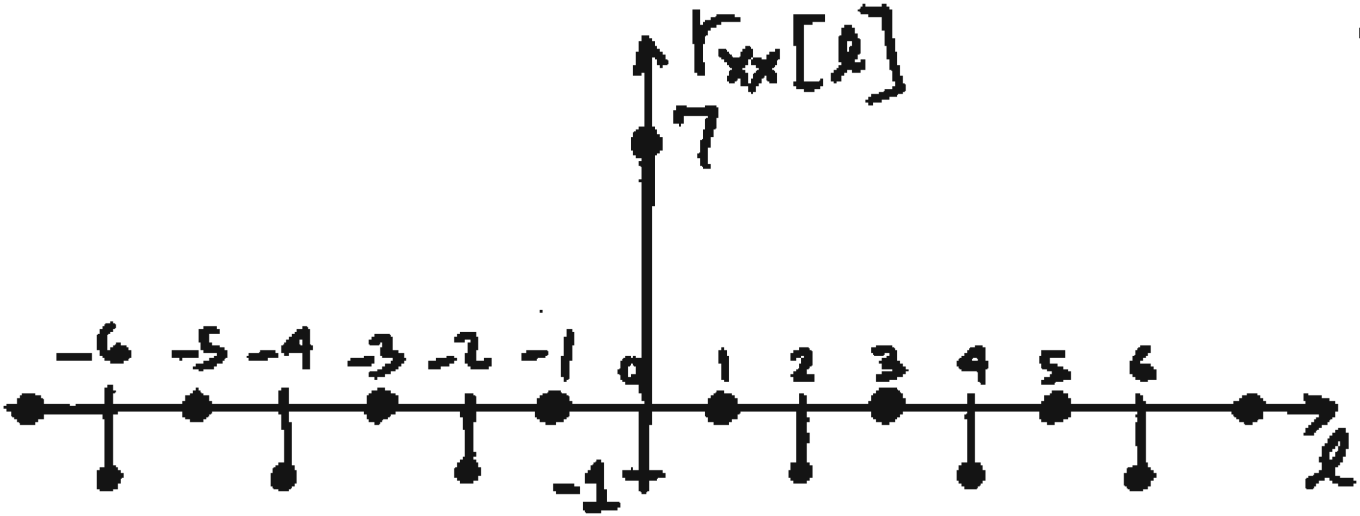
$r_{xx}[5] = 0$

$r_{xx}[6] = -1$

$r_{xx}[\ell] = 0$ for $|\ell| > 6$

$r_{xx}[-\ell] = r_{xx}[\ell]$

$\Rightarrow$ even function



- example of a Barker code
- sharp peak at $\ell = 0$
- Time-Delay Estimation in Radar
- transmit pulse: $s_a(t)$
- see Fig. 2.38 on pg. 119

- received "echo" after reflection off object

$$y_a(t) = \Gamma\, s_a(t-\underbrace{\tau_d}_{\text{round-trip time-delay}}) + \underbrace{w_a(t)}_{\text{noise}}$$

unknown amplitude

- sampled version:

$$y[n] = \Gamma\, s_a(nT_s - \tau_d) + w[n]$$

- assume sampling high enough
  $\tau_d = D T_s$, where D is integer

$$s_a(nT_s - DT_s) = s_a((n-D)T_s)$$

$$= s[n-D], \quad \text{where: } s[n] = s_a(nT_s)$$

- DT model:

$$y[n] = \Gamma' s[n-D] + w[n]$$

$$\tau_d = DT_s = \frac{2R}{c}$$ R: range to target, c = speed of light

- use cross-correlation to estimate D

$$\Rightarrow R = \frac{cDT_s}{2}$$

$$r_{ys}(\ell) = \sum_{n} y[n] \underbrace{s[n-\ell]}_{\text{stored in memory}}$$

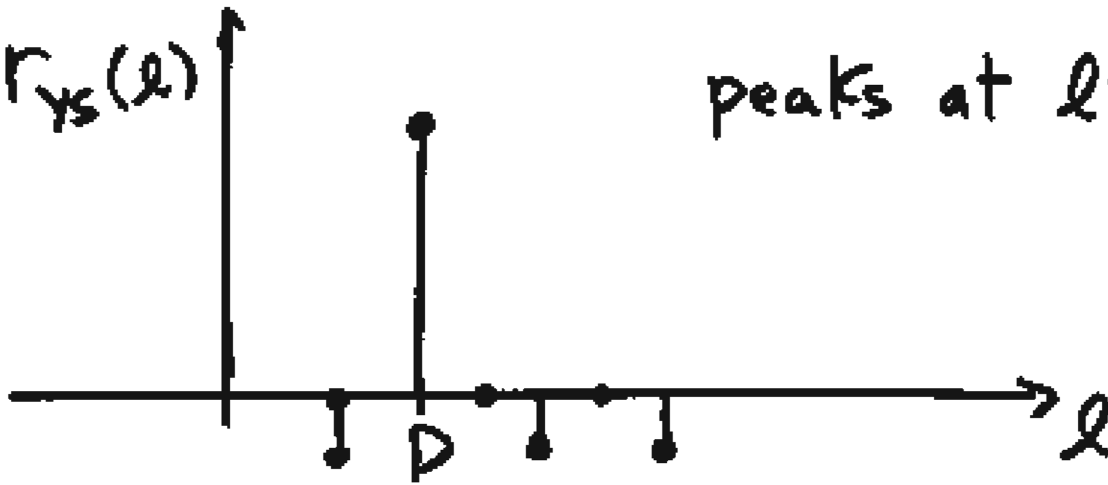
$$= \sum_{n=-\infty}^{\infty} \left(T s[n-D] + w[n] \right) s[n-\ell]$$

$$= T \sum_{n=-\infty}^{\infty} s[n-D]\, s[n-\ell] + \sum_{n=-\infty}^{\infty} w[n] s[n-\ell]$$

- change of variables: $n' = n - D$
$\Rightarrow n = n' + D$

$$= T \sum_{n'=-\infty}^{\infty} s[n'] s[n' - (\ell - D)] + r_{ws}[\ell]$$

$$r_{ys}(\ell) = r_{ss}[\ell - D] + r_{ws}[\ell]$$



- HmwK: Prob. 2.62 in Text
  $\Rightarrow$ use Matlab
- Due: Session TBD
- See cdmaeg.m at web site

Properties of Auto-Correlation

(i) $r_{xx}[-\ell] = r_{xx}^*[\ell]$

(ii) $|r_{xx}[\ell]| \leq r_{xx}[0]$

(iii) $\sum_{\ell=-\infty}^{\infty} r_{xx}[\ell]\, e^{-j\omega\ell} > 0$ for all $\forall \omega$

and is real-valued

(iii) easy to deduce from

$$r_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

Also: $r_{yx}[-\ell] = r_{xy}^*[\ell]$

- A useful result $\Rightarrow$ Example 2.6.2

$$x[n] = a^n u[n]$$

- the autocorrelation sequence is

$$r_{xx}[\ell] = \frac{1}{1-a^2} a^{|\ell|}$$

a real-valued and $0 < a < 1$

- could show up in Exam (Open Book)

I/O Relationships for LTI System

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow y[n]$$

$$r_{yx}[\ell] = r_{xx}[\ell] * h[\ell]$$

cross-correlation between input and output

$$r_{yy}[\ell] = r_{xx}[\ell] * r_{hh}[\ell]$$

Additional results for Autocorrelation

1. $x[n]$ and $y[n] = x[n-n_0]$ have the same autocorrelation function: $r_{yy}[\ell] = r_{xx}[\ell]$

Proof:

$$x[n] \longrightarrow \boxed{h[n] = \delta[n-n_0]} \longrightarrow y[n] = x[n-n_0]$$

$$= x[n] * \delta[n-n_0]$$

- From previous, we have I/O relationship for LTI System:

$$r_{yy}[\ell] = r_{xx}[\ell] * h[\ell] * h^*[-\ell]$$

- applied here with $h[n] = \delta[n-n_0]$:

$$r_{yy}[\ell] = r_{xx}[\ell] * \delta[\ell-n_0] * \delta[-\ell-n_0]$$

since $\delta[n]$ is symmetric

$$= r_{xx}[\ell] * \delta[\ell-n_0] * \delta[\ell+n_0]$$

$$= r_{xx}[\ell]$$ since $\delta[\ell-n_0] * \delta[\ell+n_0] = \delta[\ell]$ and $r_{xx}[\ell] * \delta[\ell] = r_{xx}[\ell]$

2. Autocorrelation for

$$y[n] = e^{j(\omega_o n+\theta)} x[n]$$

is: $r_{yy}[\ell] = e^{j\omega_o \ell} r_{xx}[\ell]$

Proof: $r_{yy}[\ell] = y[\ell] * y^*[-\ell]$

$$= e^{j\theta} e^{j\omega_o \ell} x[\ell] * e^{-j\theta} e^{-j\omega_o(-\ell)} x^*[-\ell]$$

$$= \{e^{j\omega_o \ell} x[\ell]\} * \{e^{j\omega_o \ell} x^*[-\ell]\}$$

$$= \sum_k e^{j\omega_o k} x[k]\, e^{j\omega_o(\ell-k)} x^*[-(\ell-k)]$$

$$= e^{j\omega_o \ell} \sum_k x[k]\, x^*[-(\ell-k)]$$

$$= e^{j\omega_o \ell} r_{xx}[\ell]$$