EE538 DSP I

Module 19

Outline:

- Further matlab demos on FIR Differentiator Design — Sect. 8.2.5
- Hilbert Transform Design via Remez-Exchange Algorithm — 8.2.6
  - matlab demo: hilberteg.m
- Intro to Discrete Fourier Transform

- Design of Differentiators
- Background in CT:

$$x_a(t) \longrightarrow \boxed{H(s) = s} \longrightarrow \frac{d\,x_a(t)}{dt}$$

$$H(\Omega) = H(s)\Big|_{s=j\Omega} = j\Omega$$

- in Hz: $H(F) = j2\pi F$

$$\frac{d\,x_a(t)}{dt} \xleftrightarrow{\text{CTFT}} j2\pi F\, X_a(F)$$

- in DT:

$$x[n] \longrightarrow \boxed{H(\omega) = j\omega} \longrightarrow y[n]$$

$$x[n] = x_a(nT_s) \qquad y[n] = \left.\frac{d\,x_a(t)}{dt}\right|_{t=nT_s}$$

- note:

only require:

$$H(\omega) = j\omega \quad \text{for } |\omega| < 2\pi \frac{W}{F_s}$$

$$|H(\omega)| = |\omega| \qquad \angle H(\omega) = \begin{cases} \frac{\pi}{2}, & 0 < \omega < \pi \\ -\frac{\pi}{2}, & -\pi < \omega < 0 \end{cases}$$

$$j = e^{j\pi/2}$$

- in this case, must employ anti-symmetric filter where

$$h[n] = -h[M-1-n],$$

$$n = 0, 1, \ldots, M-1$$

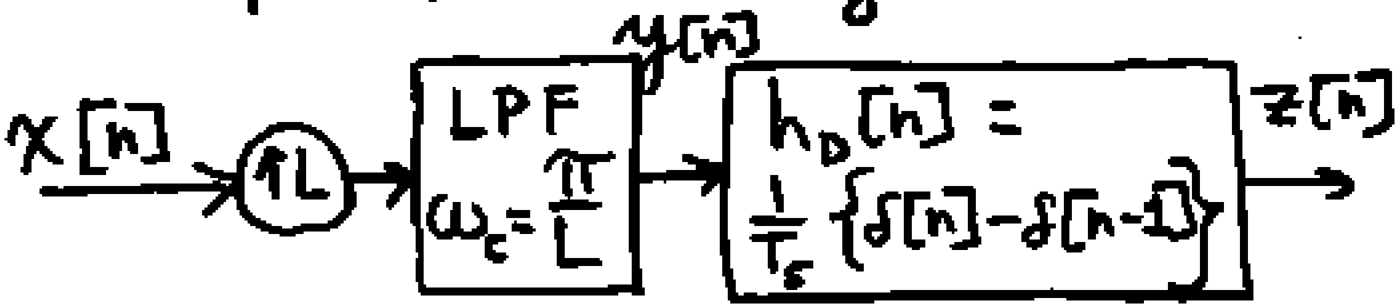
- can show:

$$H(\omega) = j \underbrace{H_r(\omega)}_{\text{purely real-valued but possibly negative}} e^{-j\frac{(M-1)}{2}\omega}$$

- in matlab:

remez(M, F, A, W, 'differentiator')

• alternative approach to differentiation: digital upsampling followed by simple differencing



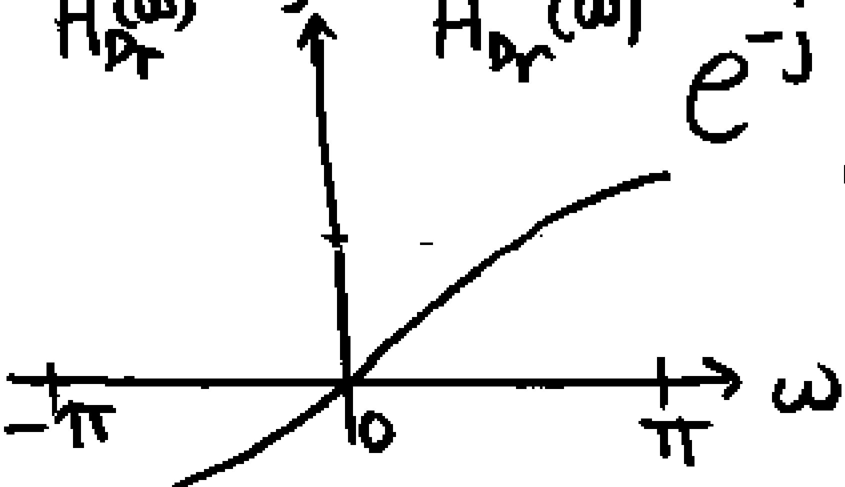
$$T_s' = \frac{T_s}{L} \qquad z[n] = \frac{1}{T_s'}\{y[n] - y[n-1]\}$$

• DTFT of $h_D[n]$

$$H_D(\omega) = \frac{1}{T_s'}\{1 - e^{-j\omega}\}$$

$$H_D(\omega) = \frac{e^{-j\frac{\omega}{2}}}{T_s'}\left\{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right\}$$

$$= \frac{2}{T_s'}\, j\, \underbrace{\sin\left(\frac{\omega}{2}\right)}_{H_{Dr}(\omega)}\ \underbrace{e^{-j\frac{\omega}{2}}}_{e^{-j\frac{M-1}{2}\omega}}$$



where $M = 2$.

Recall ideally:

$H_{Diff}(\omega) = j\omega$

- for $\omega \to 0$, $\sin\left(\frac{\omega}{2}\right) \approx \frac{\omega}{2}$
- thus: $H_D(\omega) \approx j\omega$ for $\omega \ll 1$
- see demo derivegz.m
  at course web site

- Digital Hilbert Transformers
- motivation:
- if $x[n]$ is real-valued,

$$|X(-\omega)| = |X(\omega)| \quad \text{magnitude even}$$

$$\angle X(-\omega) = -\angle X(\omega) \quad \text{phase odd}$$

- for transmission purposes, spectral info. in $-\pi < \omega < 0$ is redundant

- transmit more info. per unit bandwidth by only transmitting spectral info. in $0 < \omega < \pi$
- can "blank out" spectral info. in $-\pi < \omega < 0$ by creating complex-valued "analytic" signal

$$\tilde{x}[n] = x[n] + j\,\hat{x}[n]$$

- where: $\hat{x}[n] = x[n] * h_{HT}[n]$

$\hat{x}[n]$ is Hilbert Transform of $x[n]$

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h_{HT}[n]\end{matrix}} \longrightarrow \hat{x}[n]$$

- in freq. domain: $\hat{X}(\omega) = H_{HT}(\omega)\, X(\omega)$
- where ideally:

$$H_{HT}(\omega) = \begin{cases} -j, & 0 < \omega < \pi \\ +j, & -\pi < \omega < 0 \end{cases}$$

- Why? Observe:

$$\tilde{x}[n] = x[n] + j\, x[n] * h_{HT}[n]$$
$$= x[n] * \underbrace{\left\{ \delta[n] + j\, h_{HT}[n] \right\}}_{\tilde{h}}$$
$$= x[n] * \tilde{h}[n]$$

- in freq. domain: $\tilde{X}(\omega) = X(\omega)\tilde{H}(\omega)$
- where:

$$\tilde{H}(\omega) = 1 + jH_{HT}(\omega) = \begin{cases} 1 + j(-j) = 2 & \text{for } 0 < \omega < \pi \\ 1 + j(j) = 0 & \text{for } -\pi < \omega < 0 \end{cases}$$

- need infinite length filter to achieve discontinuous jump at $\omega = 0$ (from $-j$ to $+j$)
- in practice => consider human speech, for example:
- negligible energy below 300 Hz
- so for $F_s = 12$ KHz:

$$\omega_{p0} = \frac{.3}{12}(2\pi) = \frac{\pi}{20}$$

$$\omega_{p1} = \frac{4}{12}(2\pi) = \frac{2\pi}{3}$$

- require:

$$H_{HT}(\omega) = \begin{cases} -j, & \omega_{p0} < \omega < \omega_{p1} \\ +j, & -\omega_{p1} < \omega < -\omega_{p0} \end{cases}$$

- net result:

$$\tilde{X}(\omega) = 0 \text{ for } -\pi < \omega < 0$$

- see demo hilberteg.m at web site

- $\tilde{H}(\omega) = \begin{cases} 2, & 0<\omega<\pi \\ 0, & -\pi<\omega<0 \end{cases}$ period still $= 2\pi$
- thus: $\tilde{X}(\omega)=0$ for $-\pi<\omega<0$

- need infinite length filter to achieve discontinuity at $\omega=0$
- Note: $H_{HT}(\omega)$ is purely imaginary
- for FIR filter design, require $h[n]$ to be anti-symmetric as with differentiator.

• recall: if $h[M-1-n] = -h[n]$

$n = 0, 1, \ldots, M-1$

• $H(\omega) = j \underbrace{H_r(\omega)}_{\text{purely-real-valued with}} e^{-j\frac{M-1}{2}\omega}$

• $H_r(\omega) = 0$ for $\omega = 0$ for M odd & M even

• for M odd:

$$H_r(\omega) = 2\sum_{n=0}^{(M-1)/2} h[n] \sin\left[\left(\frac{M-1}{2} - n\right)\omega\right]$$

• if M is odd, $\frac{M-1}{2}$ is an integer

• thus:

$$H_r(\pi) = 2\sum_{n=0}^{(M-1)/2} h[n] \sin\Big[\Big(\underbrace{\tfrac{M-1}{2} - n}_{\text{integer}}\Big)\pi\Big]$$

$$= 0$$

• for M even:

$$H_r(\omega) = 2\sum_{n=0}^{M/2-1} h[n] \sin\Big[\Big(\frac{M-1}{2} - n\Big)\omega\Big]$$

$$H_r(\pi) \neq 0$$

• this is significant for both differentiator and Hilbert Transform design

- for differentiator:

$$H(\omega)\Big|_{\omega=\pi} = j\pi \neq 0$$

- for Hilbert Transformer:

$$H(\omega)\Big|_{\omega=\pi} = -j \neq 0$$

- thus, to use M odd, must oversample original analog signal so that there is a don't care transition region
- M even preferred but $(M-1)/2$ is not integer delay