EE538 DSP I

Module 18

Outline:

- Optimum (Equi-Ripple) Linear Phase FIR Filter Design
  - Sect. 8.2.4 – see notes at web site
- lowpass/bandpass examples
- differentiator example – Sect. 8.2.5
- Hilbert Transform example – 8.2.6

Development on next slide picks up where Module 16 finished.
There is no Module 17.

- See Sect. 8.2.4 for following development related to the Parks-McClellan Algorithm

$$H_r(\omega) = \sum_{k=0}^{(M-1)/2} \alpha[k] \cos(k\omega)$$

· where:

$$\alpha[k] = \begin{cases} h\left[\frac{M-1}{2}\right], & k=0 \\ 2h\left[\frac{M-1}{2}-k\right], & k=1, \ldots, \frac{M-1}{2} \end{cases}$$

· For M even, see Text Table 8.5 on pg. 641

- from this point onwards, we'll solve for $\alpha(k) \Rightarrow$ determine $h(n)$ from $\alpha(k)$ via this relationship

ILPF Design: Desired Response

$$H_d(\omega) = H_{dr}(\omega)\, e^{-j\left(\frac{M-1}{2}\right)\omega}$$

$$H_{dr}(\omega) = \begin{cases} 1, & 0 \le \omega \le \omega_p \\ 0, & \omega_s \le \omega \le \pi \end{cases}$$

- error:

$$E(\omega) = W(\omega)\left\{H_{dr}(\omega) - H_r(\omega)\right\}$$

- for LPF design:

$$W(\omega) = \begin{cases} \delta_2/\delta_1, & 0 \le \omega \le \omega_p \\ 1, & \omega_s < \omega \le \pi \end{cases}$$

- employing this $W(\omega)$, select M to achieve δ_2 in stopband via empirical formula in text

Parks/McClellan formulated filter design problem as a minimax optimization problem

$$\underset{\{\alpha(k)\}}{\text{Minimize}} \left\{ \max_{\omega \in S'} |E(\omega)| \right\}$$

- S' = passband(s) $\cup$ stopband(s)
- for LPF: $S' = \{(0, \omega_p) \cup (\omega_s, \pi)\}$

$$\min_{\{\alpha(k)\}} \left\{ \max_{\omega \in S} \left| W(\omega)\left\{ H_{dr}(\omega) - \sum_{k=0}^{L} \alpha(k)\cos(k\omega) \right\} \right| \right\}$$

where: $L = (M-1)/2$

- Solve via Chebyshev approximation Theory
- See Alternation Theorem on pg. 643

- ω_i: extremal frequency at which $H_r(\omega)$ meets error tolerance

$$W(\omega_i)\{H_{dr}(\omega_i) - H_r(\omega_i)\} = \pm\delta$$

- where $W(\omega)$ defined previously, $\delta = \delta_2$

$$E(\omega_{i+1}) = -E(\omega_i)$$

- alternation theorem dictates

· where: $\omega_1 < \omega_2 < \ldots < \omega_{L+2}$

$$P(\omega) = H_r(\omega) = \sum_{k=0}^{L} \alpha(k) \cos(k\omega)$$

· note: $\cos^2(\omega) = \frac{1}{2} + \frac{1}{2}\cos(2\omega)$

$$\cos(2\omega) = 1 - 2\cos^2(\omega)$$

$$\cos^3(\omega) = \cos(\omega)\cos^2(\omega)$$

$$\cos(3\omega) = -3\cos(\omega) + 4\cos^3(\omega)$$

$$\cos(k\omega) = \sum_{n=0}^{k} \beta_{nk} (\cos w)^n$$

$$P(\omega) = H_r(\omega) = \sum_{k=0}^{L} \alpha(k) \left\{ \sum_{n=0}^{k} \beta_{nk} (\cos w)^n \right\}$$

$\alpha(k)$

$$= \sum_{k=0}^{L} \alpha'(k) (\cos w)^k$$

let $x = \cos \omega$

- take derivative wrt ω:

$$\frac{d}{d\omega} H_r(\omega) = \frac{d}{d\omega}\left\{\sum_{k=0}^{L} \alpha'(k)\, x^{k}\right\}$$

$$= \underbrace{\left\{\sum_{k=1}^{L} \alpha'(k)\, k\, x^{k-1}\right\}}_{\text{polynomial of order } L-1 \Rightarrow \text{ has at most } L-1 \text{ roots}} \frac{dx}{d\omega}$$

$$\frac{dx}{d\omega} = \frac{d}{d\omega}(\cos\omega) = -\sin(\omega)$$

$$= 0 \quad \text{at} \quad \omega = 0 \text{ and } \omega = \pi$$

- also: ω_p and ω_s are possible extremal frequencies
- thus, at most $L+3$ extremal frequencies

- alternation theorem requires at least $L+2$ extremal freqs.

$$W(\omega_i)\left\{ \underset{\substack{\uparrow \\ H_{dr}(\omega_i)}}{\cancel{\ast}} - \sum_{k=0}^{L} \alpha(k)\cos(k\omega_i) \right\} = (-1)^i \delta$$

$i = 1, 2, \ldots, L+2$

- Know neither $\{\omega_i\}$ or $\{\alpha(k)\}$ or δ
- Use Remez-Exchange Algorithm to solve for both

Summary:

1. Guess at $L+2$ extremal freqs.
2. Solve for $\{\alpha(\ell)\}$ and δ
   - construct $H_r(\omega)$ and $E(\omega)$
3. if $|E(\omega)| < \delta$ for all $\omega \Rightarrow$ STOP
4. update $\{\omega_i\}$ as $L+2$ extremal frequencies of new $E(\omega)$
5. go to step 2.

$$\sum_{k=0}^{L} \alpha(k)\cos(k\omega_i) + \frac{(-1)^i \delta}{W(\omega_i)} = H_{dr}(\omega_i)$$

$$i = 1, 2, \ldots, L+2$$

$$\begin{bmatrix} 1 & \cos\omega_1 & \cdots & \cos(L\omega_1) & \frac{-1}{W(\omega_1)} \\ 1 & \cos\omega_2 & \cdots & \cos(L\omega_2) & \frac{1}{W(\omega_2)} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \cos(\omega_{L+2}) & \cdots & \cos(L\omega_{L+2}) & \frac{-1}{W(\omega_{L+2})} \end{bmatrix} \begin{bmatrix} \alpha(0) \\ \alpha(1) \\ \vdots \\ \alpha(L) \\ \delta \end{bmatrix} = \begin{bmatrix} H_{dr}(\omega_1) \\ H_{dr}(\omega_2) \\ \vdots \\ H_{dr}(\omega_{L+2}) \end{bmatrix}$$

- rule of thumb for doing steps 3 and 4:
- evaluate $E(\omega)$ at $16M$ equi-spaced frequencies in stopband and passband
- show set of eqns. need to be solved in Step 2 for a given set $\{\omega_i\}$, $i=1,\ldots,L+2$

- Parks/McClellan algorithm requires user specify filter length M and only guarantees ratio of stopband to passband ripple is δ_2/δ_1 ⇒ NOT that passband ripple is δ_1 and stopband ripple is δ_2
- See pg. 664 for empirical formula eqns. (8.2.95) – (8.2.97)

- Design of Differentiators
- Background in CT:

$$x_a(t) \longrightarrow \boxed{H(s) = s} \longrightarrow \frac{d\,x_a(t)}{dt}$$

$$H(\Omega) = H(s)\Big|_{s=j\Omega} = j\Omega$$

- in Hz: $H(F) = j2\pi F$

$$\frac{d\,x_a(t)}{dt} \overset{CTFT}{\longleftrightarrow} j2\pi F\, X_a(F)$$

- in DT:

$$x[n] \longrightarrow \boxed{H(\omega) = j\omega} \longrightarrow y[n]$$

$= x_a(nT_s)$ $\qquad$ $= \left. \dfrac{d x_a(t)}{dt} \right|_{t = nT_s}$

- note:

only require:

$$H(\omega) = j\omega \text{ for } |\omega| < 2\pi \frac{W}{F_s}$$

$$|H(\omega)| = |\omega| \qquad \angle H(\omega) = \begin{cases} \frac{\pi}{2}, & 0 < \omega < \pi \\ -\frac{\pi}{2}, & -\pi < \omega < 0 \end{cases}$$

$$j = e^{j\pi/2}$$

- in this case, must employ anti-symmetric filter where

$$h[n] = -h[M-1-n], \quad n = 0, 1, \dots, M-1$$

- can show:

$$H(\omega) = j \underbrace{H_r(\omega)}_{\substack{\text{purely real-valued} \\ \text{but possibly negative}}} e^{-j\frac{(M-1)}{2}\omega}$$

- in matlab:

remez(M, F, A, W, 'differentiator')