EE538 DSP I

Module 14

Outline:

- Characteristics of Some Common Analog Filter Design Techniques – Sect. 8.3.5
- Bilinear Transform Method of Design IIR Filters - Sect. 8.3.3

- $$x_a(t) \longrightarrow \boxed{\begin{array}{c}\text{Analog}\\ \text{Filter}\\ h_a(t)\end{array}} \longrightarrow y_a(t)$$

$$\mathcal{L}\{h_a(t)\} = H_a(s) = \frac{Y_a(s)}{X_a(s)}$$

$$= \frac{\prod_{k=1}^{M} (s - z_k)}{\prod_{k=1}^{N} (s - p_k)}$$

$$H_a(s) = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=1}^{N} \alpha_k s^k} \qquad \alpha_N = 1$$

- $s = \sigma + j\Omega$

- map from s-plane to z-plane ($z = re^{j\omega}$)

$$H(z) = H_a(s) \Big|_{s = \frac{n(z)}{d(z)}}$$

- rational $H_a(s)$ leads to rational $H(z)$ which may be implemented as a difference equation
- mapping should also possess the following properties

1. The $j\Omega$ axis in s-plane mapped onto (1-to-1) unit circle in z-plane

- guarantees equi-ripple (or monotonic decreasing) property preserved thru transformation
- note: 1-to-1 mapping between each F in $-\infty < F < \infty$ and each ω in $-\pi < \omega < \pi$

2. LHP of s-plane mapped into inside of unit circle

- guarantees poles in LHP of s-plane mapped into poles inside unit circle in z-plane
- stable analog filter mapped to stable digital filter
- bilinear transform:

$$s = c\,\frac{z-1}{z+1}$$

constant

- investigate if mapping satisfies two desired properties

$$s = \sigma + j\Omega \implies z = re^{j\omega}$$

$$\sigma + j\Omega = c\,\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\left(\frac{re^{-j\omega} + 1}{re^{-j\omega} + 1}\right)$$

$$= c\,\frac{(r^2 - 1) + j2r\sin(\omega)}{r^2 + 2r\cos\omega + 1}$$

• equating real + imaginary parts on both sides of eqn.:

$$\sigma = c \, \frac{r^2 - 1}{r^2 + 2r\cos\omega + 1}$$

$$\Omega = c \, \frac{2r\sin\omega}{r^2 + 2r\cos\omega + 1}$$

• note: $r^2 + 2r\cos\omega + 1 > 0$

- observe:

$\sigma < 0$ (LHP) dictates

$r^2 - 1 < 0 \Rightarrow r^2 < 1 \Rightarrow r < 1$

- LHP of s-plane mapped into inside of unit circle

- consider $\sigma = 0 \Rightarrow$ implies $r = 1$

$\Rightarrow$ $j\Omega$-axis in s-plane is mapped to unit circle in z-plane

• setting $r = 1$ in 2^{nd} expression:

$$\Omega = c \frac{2 \sin\omega}{2 + 2\cos\omega} = c \frac{\sin\omega}{1 + \cos\omega}$$

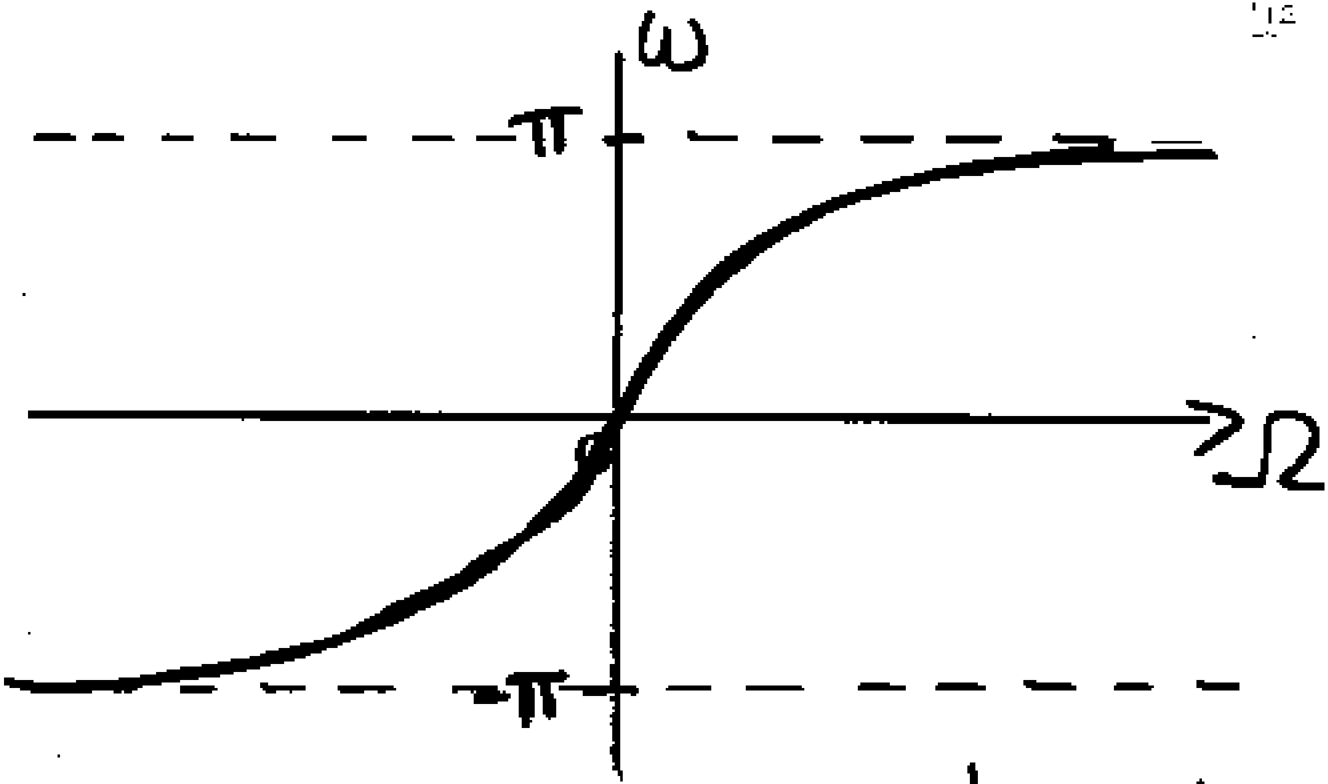
• $\cos^2\left(\frac{\omega}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos(\omega)$

• $\sin(\omega) = 2 \sin\left(\frac{\omega}{2}\right)\cos\left(\frac{\omega}{2}\right)$

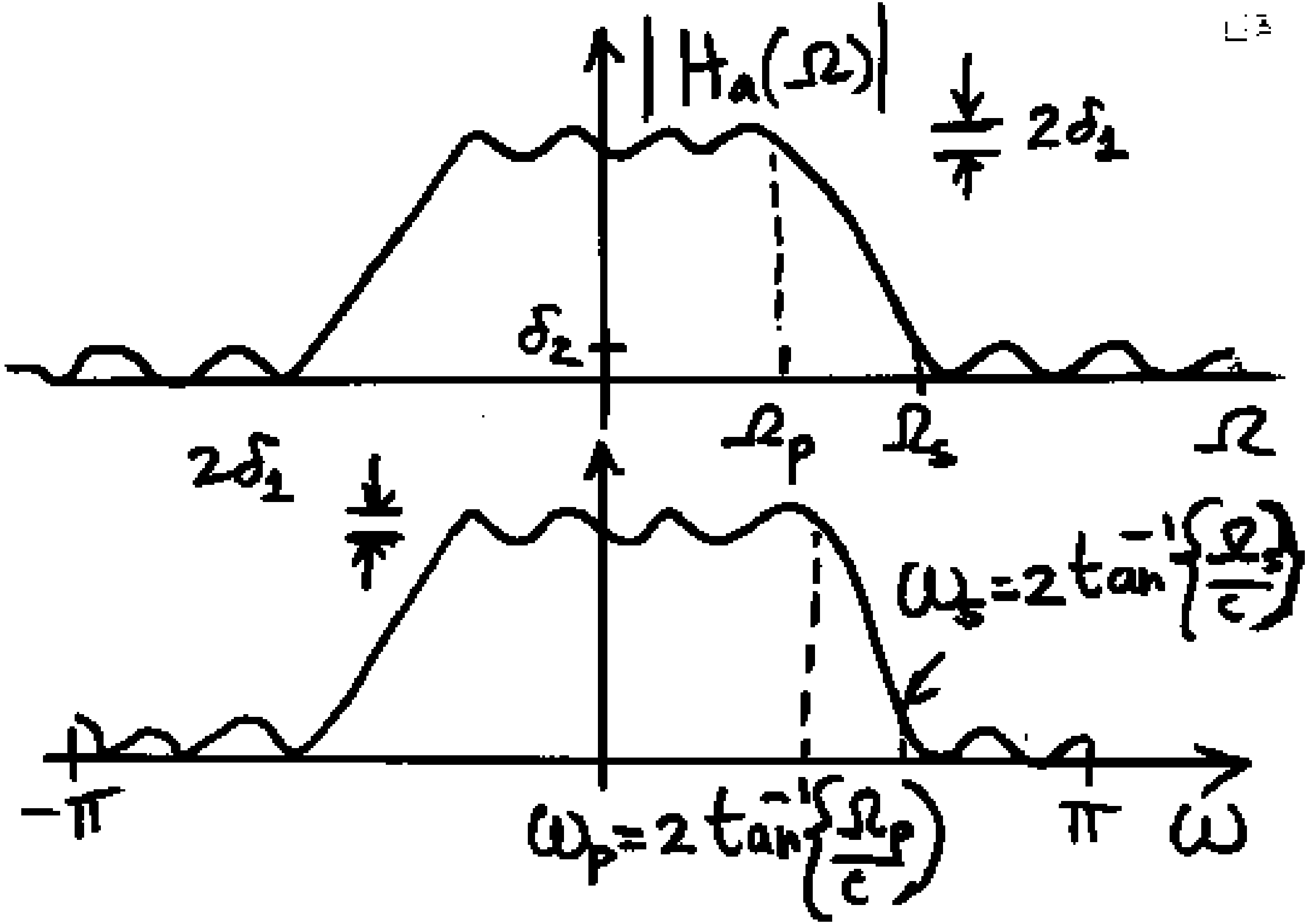
$$\Omega = c \frac{2 \sin\left(\frac{\omega}{2}\right)\cos\left(\frac{\omega}{2}\right)}{2 \cos^2\left(\frac{\omega}{2}\right)}$$

$$\Omega = c\,\frac{\sin\left(\frac{\omega}{2}\right)}{\cos\left(\frac{\omega}{2}\right)} = c\,\tan\left(\frac{\omega}{2}\right)$$

$$\Rightarrow \boxed{\omega = 2\tan^{-1}\left\{\frac{\Omega}{c}\right\}}$$



- 1-to-1 mapping as desired
- mapping is nonlinear
- frequency compression (warping)



$|H_a(\Omega)|$
$2\delta_1$
δ_2
Ω_p
Ω_s
Ω
$2\delta_1$
$\omega_s = 2\tan^{-1}\left(\frac{\Omega_s}{c}\right)$
$-\pi$
$\omega_p = 2\tan^{-1}\left(\frac{\Omega_p}{c}\right)$
π
ω

- Design procedure:

1. Transform design specs in digital domain to analog specs:

$$\Omega_p = c \tan\left(\frac{\omega_p}{2}\right) \; ; \; \Omega_s = c \tan\left(\frac{\omega_s}{2}\right)$$

$$\delta_{1a} = \delta_1 \quad ; \quad \delta_{2a} = \delta_2$$

2. Design analog LPF to meet these specs.

3. $H(z) = H_a(s)\Big|_{s = c\,\frac{z-1}{z+1}}$

4. Convert to difference eqn.

- Simple Illustrative Example

- design spec: single pole LPF with 3 dB-cutt-off at $\omega_c = .2\pi$
- 1^{st}-order Butterworth filter
  $H_a(s) = \Omega_c / s + \Omega_c$

$$\Omega_c = c \tan\left(\frac{.2\pi}{2}\right) = .325c$$

$$H_A(z) = \left.\frac{.325c}{s+.325c}\right|_{s=c\frac{z-1}{z+1}}$$

$$= \frac{.325\cancel{c}}{\cancel{c}\frac{z-1}{z+1}+.325\cancel{c}} = \frac{.325(z+1)}{z-1+.325(z+1)}$$

$$= \frac{.325(z+1)}{1.325z-.675} = \frac{.245(1+z^{-1})}{1-.509z^{-1}}$$

$$y[n] = .509\, y[n-1] + .245\, x[n] + .245\, x[n-1]$$