EE538 — Final Exam — Fall 1999
Digital Signal Processing I — Live: 16 December 1999

Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains **six** problems.
All work should be done in the blue books provided.
You must show all work for each problem to receive full credit.
Do **not** return this test sheet, just return the blue books.

Prob. No.	**Topic(s) of Problem**	**Points**
1.	Sampling, Discrete-Time Modelling of CT Systems	30
2.	Pole-Zero Frequency Analysis of LTI Systems	30
3.	FIR Filter Design	30
4.	Windows	40
5.	Sum of Sinewaves (Harmonic Spectral Analysis)	30
6.	Autoregressive Spectral Analysis	40

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Problem 1. [30 points] Consider the transmission of a pulse amplitude-modulated signal described by

$$x(t) = \sum_{k=-\infty}^{\infty} b[k]p(t - kT_o)$$

where $b[n]$ are the information-bearing symbols being transmitted which be viewed as a discrete-time sequence. In binary phase-shift keying, $b[n]$ is either "+1" or "-1" for all n. $p(t)$ is the pulse symbol waveform and $1/T_o$ is the bit rate. For this problem, sampling $p(t)$ at TWICE the bit rate yields the discrete-time sequence

$$\tilde{p}[n] = p\left(n\frac{T_o}{2}\right) = \{0, 1, 0, -2, \underbrace{4}_{\uparrow}, -2, 0, 1, 0\}$$

At the receiver, $x(t)$ arrives by both a direct path and a multipath reflection that arrives at a delay of τ with the same strength as the direct path in-phase. The received signal, $y(t)$, may be modeled as:

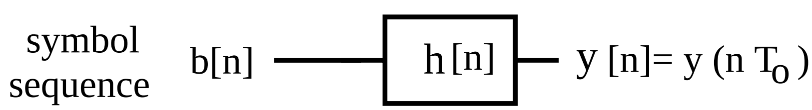
$$y(t) = x(t) * g(t)$$

where $*$ denotes continuous time convolution and

$$g(t) = \delta(t) + \delta(t - \tau) \qquad (1)$$

and $\delta(t)$ is the Dirac Delta function.

Sampling $y(t)$ at the bit rate, $F_s = \frac{1}{T_o}$, it is easily shown that the resulting sequence $y[n] = y(nT_o)$ may be modeled as having been generated by the following discrete-time system



(a) For the case of $\tau = T_o$ in $g(t)$ defined in Eqn. (1), determine the impulse response $h[n]$ above for all n so that the output $y[n]$ is $y(nT_o)$ as specified. You answer should specify the numerical values of $h[n]$.

(b) Repeat (a) for the case of $\tau = \frac{T_o}{2}$.

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Problem 2. [30 points] Consider the causal LTI described by the difference equation

$$y[n] = \frac{1}{2}y[n-1] + \frac{1}{2}x[n] - x[n-1]$$

The transfer function for this system is

$$H(z) = \left(\frac{1}{2}\right)\frac{z-2}{z-\frac{1}{2}}$$

(a) Plot the pole-zero diagram for this system.

(b) Plot the magnitude of the frequency response for this system over $-\pi < \omega < \pi$. *Hint:* Analyze $H(\omega) = \sqrt{H^*(\omega)H(\omega)}$, where $H(\omega)$ is $H(z)$ is evaluated on the unit circle.

Problem 3. [30 points]

Let $h[n], n = 0, 1, 2$, be the impulse response of an FIR filter of length $M = 3$. The frequency response of the filter is the DTFT

$$H(\omega) = \sum_{n=0}^{2} h[n]e^{-j\omega n}$$

Suppose we desire to design a LPF with passband edge, $\omega_p = \pi/2$. The design criterion for selecting the filter coefficients, {h[0], h[1], h[2]}, is to maximize the ratio of the energy in the passband to the total energy, i.e.,

$$\underset{\{h[0], h[1], h[2]\}}{Maximize} \quad \frac{\frac{1}{2\pi}\int_{-\omega_p}^{\omega_p} |H(\omega)|^2 d\omega}{\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2 d\omega}$$

where $\omega_p = \pi/2$. Determine the specific numerical values of {h[0], h[1], h[2]} that meet this design criterion, i. e., solve the above optimization problem. Clearly indicate the steps required in arriving at the solution and show all work.

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Problem 4. [40 points] Consider the following window of length $M-1$, where M is an even number.

$$w[n] = e^{j\frac{\pi}{M}n}\left\{u[n] - u[n-\frac{M}{2}]\right\} * e^{-j\frac{\pi}{M}n}\left\{u[n] - u[n-\frac{M}{2}]\right\}$$

This "new" window (which is different from the one analyzed in Problem 2 of Exam 3) is obtained as the convolution of one rectangular window of length $\frac{M}{2}$ modulated by $e^{j\frac{\pi}{M}n}$ with another rectangular window of length $\frac{M}{2}$ modulated by $e^{-j\frac{\pi}{M}n}$.

(a) Determine a closed-form expression for $w[n]$ (that is, determine a simple analytical expression for the result obtained from performing the convolution.) Sketch $w[n]$ for $n = 0, 1, ..., M-2$.

(b) Is $w[n]$ a symmetric or anti-symmetric window? Briefly justify your answer (that is, don't just guess.)

(c) Let $W(\omega)$ denote the DTFT of $w[n]$. Determine a closed-form expression for $W[\omega]$. Plot the magnitude $|W(\omega)|$ over $-\pi < \omega < \pi$ showing as much detail as possible. Explicitly point out the numerical values of the specific frequencies for which $|W(\omega)| = 0$.

(d) *Analysis of mainlobe width of* $W(\omega)$: What is the null-to-null mainlobe width of $W(\omega)$? Is the mainlobe width of $W(\omega)$ the same, larger, or smaller than the mainlobe width of the DTFT of a rectangular window of the same length, $M-1$? Briefly explain.

(e) *Analysis of peak sidelobe of* $W(\omega)$: Is the peak sidelobe of $W(\omega)$ the same, larger, or smaller than the peak sidelobe of the DTFT of a rectangular window of the same length, $M-1$? Briefly explain your answer.

(f) *Analysis of sidelobes of* $W(\omega)$: What about the sidelobes other than the peak sidelobe? That is, excluding the mainlobe and the first peak sidelobe on either side of the mainlobe, are the sidelobes of $W(\omega)$ the same, larger, or smaller than the sidelobes of the DTFT of a rectangular window of the same length, $M-1$? Briefly explain.

Problem 5. [30 points]

Let $x[n]$ be a discrete-time random process containing one real-valued sinewave plus noise as described by

$$x[n] = A\cos(\omega_0 n + \Theta) + \nu[n],$$

where the amplitude, A, and frequency, ω_0, of the sinusoid are each deterministic but unknown constants and Θ is a random variable uniformly distributed over a 2π interval. $\nu[n]$ is a white noise process with autocorrelation $r_{\nu\nu}[m] = E\{\nu[n]\nu^*[n-m]\} = \delta[m]$.

You are given the following three values of the true autocorrelation sequence $r_{xx}[m] = E\{x[n]x^*[n-m]\}$:

$$r_{xx}[0] = 3; \qquad r_{xx}[1] = 1; \qquad r_{xx}[2] = -1$$

(a) Knowing that $r_{xx}[m]$ satisfies $r_{xx}[m] = -a_1 r_{xx}[m-1] - a_2 r_{xx}[m-2] + \sigma_w^2 \delta[m]$, determine the numerical values of a_1 and a_2.

(b) Determine the numerical value of $r_{xx}[3]$.

(c) Plot the spectral density $S_{xx}(\omega) = \sum_{n=-\infty}^{\infty} r_{xx}[m]e^{-jm\omega}$ over $-\pi \leq \omega \leq \pi$.

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Problem 6. [40 points]

Suppose that the random process $x[n]$ is the output of a stable LTI system with impulse response

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 2^n & n < 0 \end{cases}$$

when the input $\nu[n]$ is a zero-mean white noise process with variance σ^2. In the following let $H(z)$ denote the Z-Transform of $h[n]$ and let

$$\hat{x}_p[n] = -\sum_{k=1}^{p} a_p[k]x[n-k]$$

denote the order p minimum mean–square linear predictor of $x[n]$ given $\{x[n-k] : 1 \leq k \leq p\}$. Let $f_p[n] = x[n] - \hat{x}_p[n]$ be the prediction error, let $E_p = \mathbf{E}\{|f_p[n]|^2\}$, and let

$$A_p(z) = 1 + \sum_{k=1}^{p} a_p[k]z^{-k}$$

denote the order p prediction error filter.

(a) Find the transfer function $H(z)$ of the system and indicate its region of convergence.

(b) Find the (true) power spectral density of $x[n]$, $S_{xx}(\omega)$.

(c) Suppose that another LTI system is placed in series with $H(z)$ having a transfer function $P(z)$. The new output is called $y[n]$. If $P(z)$ is an all-pass filter for which $|P(\omega)| = 1$ for all ω, like the system analyzed in Problem 2, find the (true) power spectral density of $y[n]$, $S_{yy}(\omega)$. Is $r_{yy}[m]$ equal to $r_{xx}[m]$? Explain your answer.

(d) For the original system $H(z)$ and the process $x[n]$ determine the cofficients, $a_2[1]$ and $a_2[2]$, of the optimum second order linear predictor. *Hint:* The answer to Part (c) can be used to make (d) very simple.