Basics of Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$X(z) = \mathcal{Z}\{x[n]\}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

$$x[n] = \mathcal{Z}^{-1}\{X(z)\}$$

Linearity: $\mathcal{Z}\{a_1 x_1[n] + a_2 x_2[n]\}$

$$= a_1 X_1(z) + a_2 X_2(z)$$

- extends for more than 3 signals

Time Shift: $x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0} X(z)$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]\, z^{-n}$$

$n' = n - n_0 \Rightarrow n = n' + n_0$

limits still $-\infty$ to $+\infty$

$$Y(z) = \sum_{n'=-\infty}^{\infty} x[n']\, z^{-(n+n_0)}$$

$$= z^{-n_0} \sum_{n'=-\infty}^{\infty} x[n']\, z^{-n'} = z^{-n_0} X(z)$$

- **Convolution Property**

$y[n] = x[n] * h[n] \overset{Z}{\longleftrightarrow} Y(z) = X(z)H(z)$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

- Take ZT of both sides and invoke linearity

$$\mathcal{Z}\{y[n]\} = \sum_{k=-\infty}^{\infty} x[k]\, \mathcal{Z}\{h[n-k]\}$$

$$Y(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k} H(z)$$

$$= X(z) H(z)$$

- Thus: $H(z) = \dfrac{Y(z)}{X(z)}$ = ratio of 2 polynomials for a difference equation LTI system

Basic Z-T Pair

$$a^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-a}$$ ROC: $|z|>|a|$

Time-Shift Property dictates

$$a^{n-1} u[n-1] \xleftrightarrow{\mathcal{Z}} z^{-1}\frac{z}{z-a} = \frac{1}{z-a}$$

$$= \frac{1}{a} a^n u[n-1]$$ ROC: same as above

- Thus, one could use same rules/procedure for partial fraction expansion that you learned for the laplace Transform

$$e^{at} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a} \Longrightarrow a^{n-1} u[n-1] \xleftrightarrow{\mathcal{Z}} \frac{1}{z-a}$$

Two Additional ZT Properties

$$n\,x[n] \xleftrightarrow{\;Z\;} -z\frac{d}{dz}X(z)$$

Proof:

$$\frac{d}{dz}\left\{ X(z) = \sum_{n=-\infty}^{\infty} x[n]\,(z^{-1})^{n} \right\}$$

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} x[n]\;n\,(z^{-1})^{n-1}\,(-1)\,z^{-2}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\,n\,(z^{-1})^{n}\,(-1)\;z\;z^{-2}$$

$$= \sum_{n=-\infty}^{\infty} n\,x[n]\;z^{-n}\left(-\frac{1}{z}\right)$$

$$-z\frac{d}{dz}\left(X(z)\right) = \sum_{n=-\infty}^{\infty} n\,x[n]\,z^{-n} \qquad \text{Q.E.D.}$$

$$y[n] = a^n x[n] \xleftrightarrow{Z} X\left(\frac{z}{a}\right) = Y(z)$$

Proof:

$$\sum_{n=-\infty}^{\infty} a^n x[n] \, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(a z^{-1}\right)^n$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{a}\right)^{-n} \qquad \text{Q.E.D.}$$

See Table 3.2 on pg. 169
Z-Transform Properties