- **Key formulas for analyzing a perfect reconstruction filter bank**

$$x[n] \longrightarrow (L\uparrow) \longrightarrow y[n]$$

Insert L-1 zeros between each successive values of $x[n]$

Time Domain: $$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$$

Frequency Domain: $$Y(\omega) = X(L\omega)$$

Decimator: $$y[n] = x[Dn]$$

$$x[n] \longrightarrow (D\downarrow) \longrightarrow$$

Frequency Domain: $$Y(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\omega - k2\pi}{D}\right)$$

- Related to the polyphase filters, we (2) encountered terms like:

$$h_\ell[n] = h[Ln+\ell] \xleftrightarrow{\text{DTFT}} H_\ell(\omega) = ?$$

- First consider: $g_\ell[n] = h[n+\ell]$

$$\Rightarrow \quad G_\ell(\omega) = e^{j\omega\ell} H(\omega)$$

- Next: $h_\ell[n] = g_\ell[Ln] = h[Ln+\ell]$

- Thus: $H_\ell(\omega) = \frac{1}{L} \sum_{k=0}^{L-1} G_\ell\left(\frac{\omega - k2\pi}{L}\right)$

Substitute: $G_\ell(\omega) = e^{j\omega\ell} H(\omega)$

3

$$H_\ell(\omega) = \frac{1}{L}\sum_{k=0}^{L-1} e^{j\frac{(\omega-k2\pi)\ell}{L}} H\left(\frac{\omega-k2\pi}{L}\right)$$

$$= \left\{\frac{1}{L}\sum_{k=0}^{L-1} e^{-j\frac{k2\pi\ell}{L}} H\left(\frac{\omega-k2\pi}{L}\right)\right\} e^{j\frac{\ell}{L}\omega}$$

In ideal case: $h[n] = \dfrac{\sin\left(\frac{\pi}{L}n\right)}{\frac{\pi}{L}n}$

in which case:

$$H_\ell(\omega) = e^{j\frac{\ell}{L}\omega} \quad \text{for } -\pi < \omega < \pi$$

corresponding to shift by fractional amount $(\ell/L)\,T_s$ back in the time domain

- An alternative way to express zero inserts:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$$

$$= \begin{cases} x\left[\frac{n}{L}\right], & n = mL, \ m \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] \longrightarrow (L\uparrow) \longrightarrow y[n]$$

Inverse CTFT:

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F t}\, dF$$

CTFT:

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi F t}\, dt$$

Scaling property:

If: $x_a(t) \overset{\mathcal{F}}{\longleftrightarrow} X_a(F)$

Then: $x_a(\alpha t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|\alpha|} X_a\left(\frac{F}{\alpha}\right)$

Inverse DTFT: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega$

DTFT:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

"Scaling" Properties: $x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$

$$x[Dn] \overset{DTFT}{\longleftrightarrow} \frac{1}{D}X\left(\frac{\omega}{D}\right) + \sum_{k=1}^{D} \frac{1}{D}X\left(\frac{\omega - k2\pi}{D}\right)$$

$$y[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = mL,\ m \text{ integer} \\ 0, & \text{otherwise} \end{cases} \overset{DTFT}{\longleftrightarrow} X(L\omega)$$