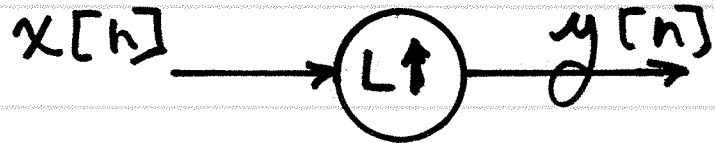


• Key formulas for analyzing a perfect reconstruction filter bank ①

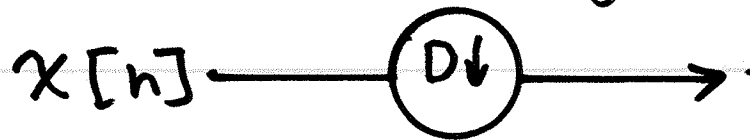


Insert $L-1$ zeros
between each successive
values of $x[n]$

Time Domain:
$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL]$$

Frequency Domain:
$$Y(\omega) = X(L\omega)$$

Decimator:
$$y[n] = x[Dn]$$



Frequency Domain:
$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega - k2\pi}{D}\right)$$

- Related to the polyphase filters, we (2) encountered terms like:

$$h_\ell[n] = h[Ln + \ell] \xleftrightarrow{\text{DTFT}} H_\ell(\omega) = ?$$

- First consider: $g_\ell[n] = h[n + \ell]$

$$\Rightarrow G_\ell(\omega) = e^{j\omega\ell} H(\omega)$$

- Next: $h_\ell[n] = g_\ell[Ln] = h[Ln + \ell]$

$$\text{Thus: } H_\ell(\omega) = \frac{1}{L} \sum_{k=0}^{L-1} G_\ell\left(\frac{\omega - k2\pi}{L}\right)$$

$$\text{Substitute: } G_\ell(\omega) = e^{j\omega\ell} H(\omega)$$

$$H_\ell(\omega) = \frac{1}{L} \sum_{k=0}^{L-1} e^{j \frac{(\omega - k2\pi)\ell}{L}} H\left(\frac{\omega - k2\pi}{L}\right) \quad (3)$$

$$= \left\{ \frac{1}{L} \sum_{k=0}^{L-1} e^{-j \frac{k2\pi\ell}{L}} H\left(\frac{\omega - k2\pi}{L}\right) \right\} e^{j \frac{\ell}{L} \omega}$$

If ideal case: $h[n] = \frac{\sin\left(\frac{\pi}{L}n\right)}{\frac{\pi}{L}n}$

in which case:

$$H_\ell(\omega) = e^{j \frac{\ell}{L} \omega} \quad \text{for } -\pi < \omega < \pi$$

corresponding to shift by fractional amount

$(\ell/L) T_s$ back in the time domain

• An alternative way to express zero inserts: ④

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL]$$

$$= \begin{cases} x\left[\frac{n}{L}\right], & n = mL, m \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$



Inverse
CTFT :

⑤

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j 2\pi F t} dF$$

CTFT:

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j 2\pi F t} dt$$

Scaling property:

$$\text{IF: } x_a(t) \xleftrightarrow{\mathcal{F}} X_a(F)$$

$$\text{Then: } x_a(\alpha t) \xleftrightarrow{\mathcal{F}} \frac{1}{|\alpha|} X_a\left(\frac{F}{\alpha}\right)$$

Inverse DTFT: $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$ (6)

DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

"Scaling" Properties: $x[n] \xleftrightarrow{\text{DTFT}} X(\omega)$

$x[Dn] \xleftrightarrow{\text{DTFT}} \frac{1}{D} X\left(\frac{\omega}{D}\right) + \sum_{k=1}^{D-1} \frac{1}{D} X\left(\frac{\omega - k2\pi}{D}\right)$

$y[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = mL, m \text{ integer} \\ 0, & \text{otherwise} \end{cases} \xleftrightarrow{\text{DTFT}} X(L\omega)$