Ideal D/A :

$$x_a(t) = \sum_{n=-\infty}^{\infty} \underbrace{x_a(nT_s)}_{x[n]} \frac{\sin\left(\frac{\pi}{T_s}(t-nT_s)\right)}{\frac{\pi}{T_s}(t-nT_s)}$$

Digital Up-Sampling :

$$y[n] = x_a\left(n\frac{T_s}{L}\right)$$ (replace summation n variable above by k)

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\frac{\pi}{T_s}\left(n\frac{T_s}{L} - kT_s\right)\right)}{\frac{\pi}{T_s}\left(n\frac{T_s}{L} - kT_s\right)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\frac{\pi}{L}(n-kL)\right)}{\frac{\pi}{L}(n-kL)}$$

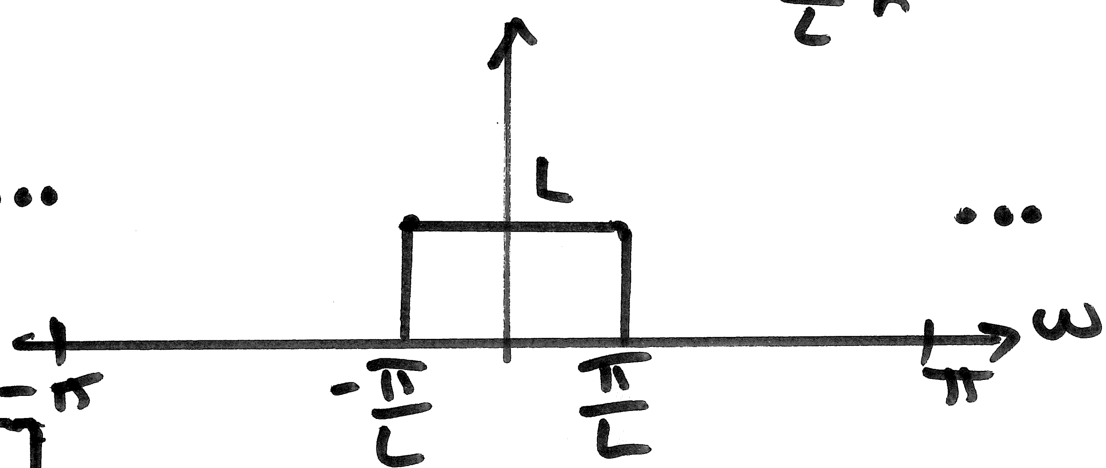
$$y[n] = \left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]\right\} * \frac{\sin\left(\frac{\pi}{L}n\right)}{\frac{\pi}{L}n}$$

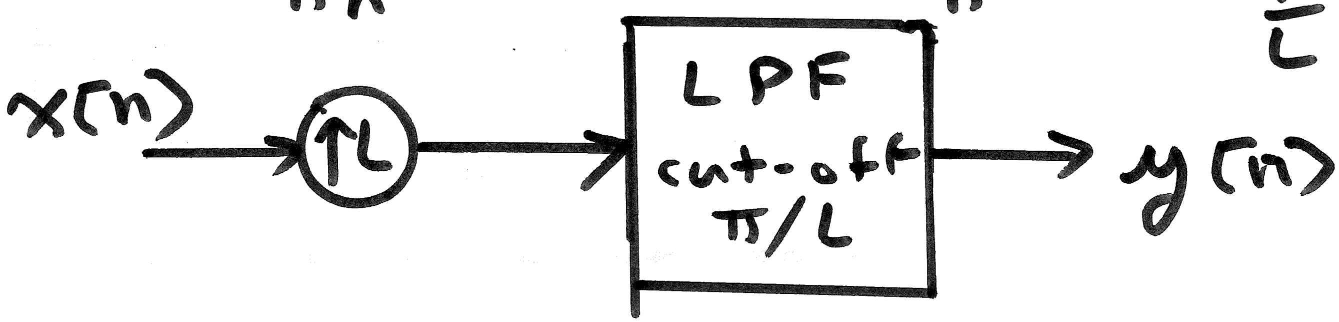
$$y[n] = x_a\left(n\frac{T_s}{L}\right) \qquad x[n] = x_a(nT_s)$$

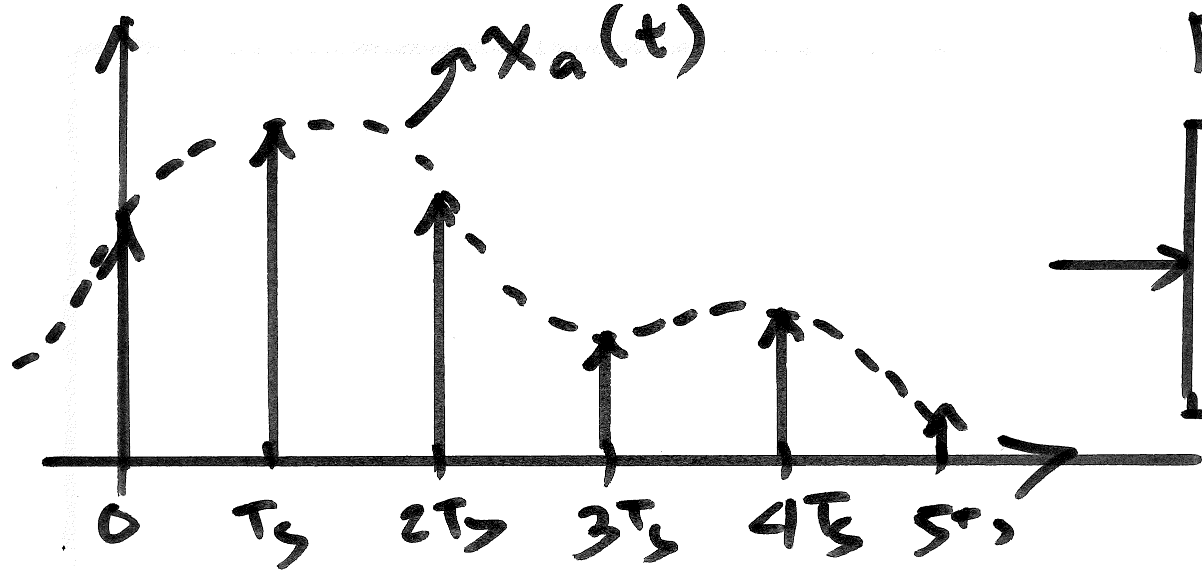
<u>Since</u>: $$\frac{\sin\left(\frac{\pi}{L}(n-kL)\right)}{\frac{\pi}{L}(n-kL)} = \delta[n-kL] * \frac{\sin\left(\frac{\pi}{L}n\right)}{\frac{\pi}{L}n}$$

From Table 4.2

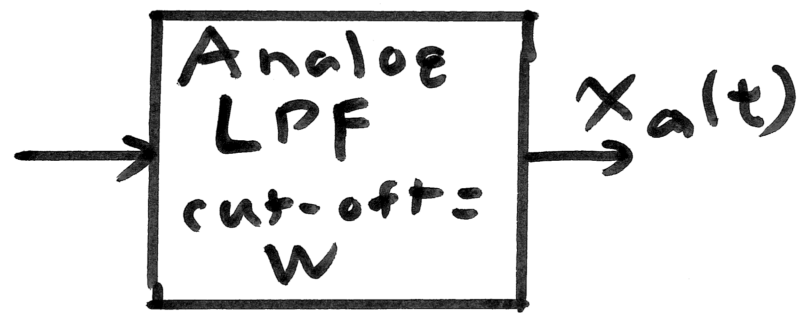
$$L\,\frac{\sin\left(\frac{\pi}{L}n\right)}{\pi n} \xleftrightarrow{\text{DTFT}} \cdots$$

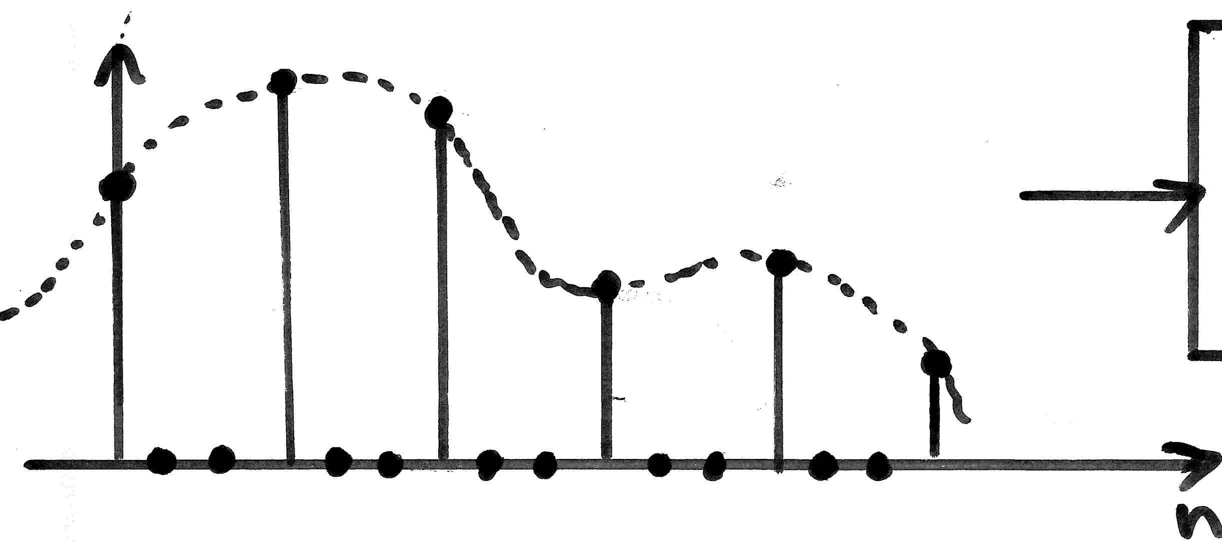


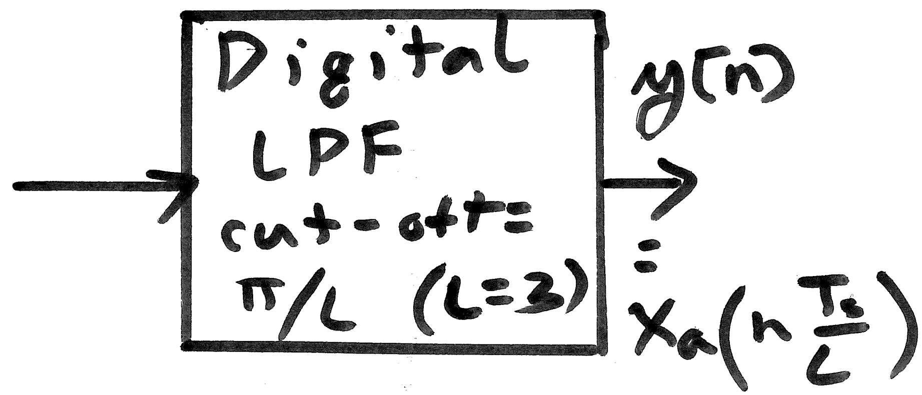




$F_s > 2W$







$L=3$
for picture

Note on Fractional Time-Shift

- Recall ideal Reconstruction Formula

$$x_a(t) = \sum_{k=-\infty}^{\infty} \underbrace{x_a(kT_s)}_{x[k]} \frac{\sin\left(\frac{\pi}{T_s}(t-kT_s)\right)}{\frac{\pi}{T_s}(t-kT_s)}$$

- Evaluate at: $t = \frac{l}{L}T_s + nT_s$ $\quad l \in \{0, 1, \ldots, L-1\}$, $L = \text{integer}$

$$y[n] = x_a\left(\frac{l}{L}T_s + nT_s\right) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\frac{\pi}{T_s}\left(\frac{l}{L}T_s + nT_s - kT_s\right)\right)}{\frac{\pi}{T_s}\left(\frac{l}{L}T_s + nT_s - kT_s\right)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\pi\left(\frac{l}{L} + n - k\right)\right)}{\pi\left(\frac{l}{L} + n - k\right)}$$

$$= x[n] * h[n] \qquad h[n] = \frac{\sin\left(\pi\left(n + \frac{l}{L}\right)\right)}{\pi\left(n + \frac{l}{L}\right)}$$

$$= x_a\left(\frac{l}{L}T_s + nT_s\right)$$

- Preamble:

$$x[n] * \delta[n-n_0] = x[n-n_0]$$

- in particular, if $n_0 = 0$:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

- Consider:

$$\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-2k]$$

$$= \dots + x[-1]\,\delta[n+2]$$
$$+ x[0]\,\delta[n]$$
$$+ x[1]\,\delta[n-2]$$
$$+ \dots$$
$$= \left\{ \dots, 0, x[-1], 0, x[0], 0, x[1], 0, \dots \right\}$$

$$x[n] \longrightarrow \uparrow 2 \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-2k]$$

- more generally:
- insert $L-1$ zeroes between ea. pair of samples

$$x[n] \longrightarrow \uparrow L \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$$

- End of Preamble