Development of Two-Channel Perfect Reconstruction Filter Bank

Sect. 11.11 P+M Text, Ed. 4

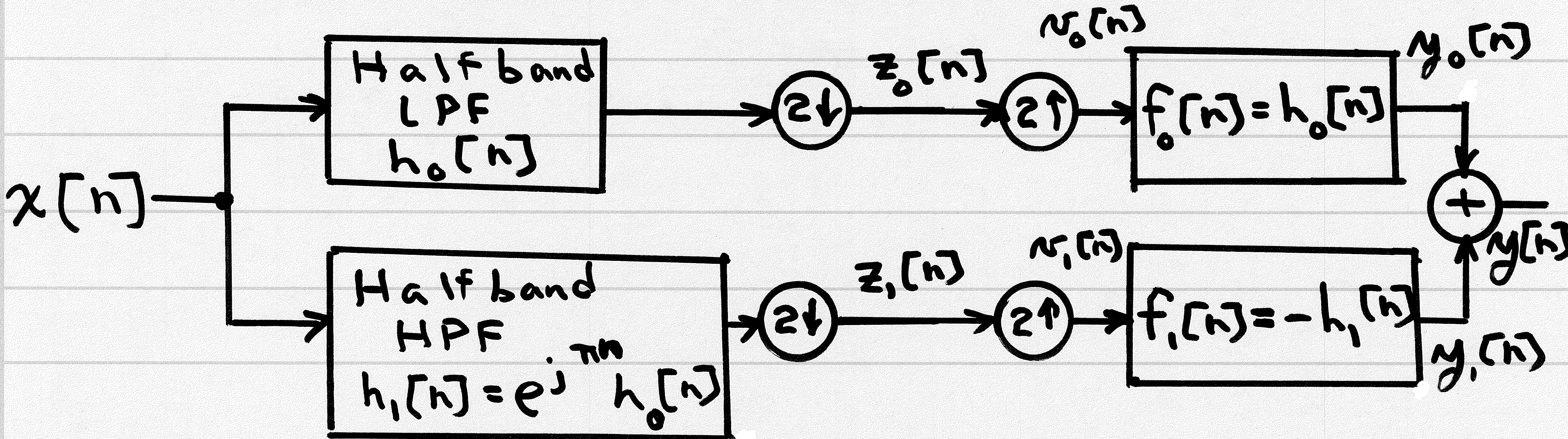
Two-Channel Quadrature Mirror Filter Bank

The following development also holds for special case of two simple half-band filters:

$$h_0[n] = \{\underset{\uparrow}{1}, 1\} \qquad h_1[n] = (-1)^n h_0[n] = \{1, -1\}$$

See page 13 (3rd from last page)

Two-Channel Perfect Reconstruction (PR) Filter Bank = Quadrature Mirror Filter (QMF)



$$Z_0(\omega) = \frac{1}{2} H_0\left(\frac{\omega}{2}\right) X\left(\frac{\omega}{2}\right) + \frac{1}{2} H_0\left(\frac{\omega - 2\pi}{2}\right) X\left(\frac{\omega - 2\pi}{2}\right)$$

$$Z_1(\omega) = \frac{1}{2} H_1\left(\frac{\omega}{2}\right) X\left(\frac{\omega}{2}\right) + \frac{1}{2} H_1\left(\frac{\omega - 2\pi}{2}\right) X\left(\frac{\omega - 2\pi}{2}\right)$$

Note: $H_1(\omega) = H_0(\omega - \pi)$ => use later

Next: $V_i(\omega) = Z_i(2\omega)$

• **Thus:**

$$V_i(\omega) = \frac{1}{2} H_i(\omega) X(\omega) + \frac{1}{2} H_i\left(\frac{2\omega - 2\pi}{2}\right) X\left(\frac{2\omega - 2\pi}{2}\right)$$

$i = 0, 1$

$$= \frac{1}{2} H_i(\omega) X(\omega) + \frac{1}{2} H_i(\omega - \pi) X(\omega - \pi)$$

See next page for matrix form of these equations

• **Continuing:** $Y(\omega) = Y_0(\omega) + Y_1(\omega)$

$$Y(\omega) = H_0(\omega)\left\{\frac{1}{2} H_0(\omega) X(\omega) + \frac{1}{2} H_0(\omega - \pi) X(\omega - \pi)\right\}$$

$$- H_1(\omega)\left\{\frac{1}{2} H_1(\omega) X(\omega) + \frac{1}{2} H_1(\omega - \pi) X(\omega - \pi)\right\}$$

Rearranging:

$$Y(\omega) = \left\{\frac{1}{2} H_0^2(\omega) - \frac{1}{2} H_1^2(\omega)\right\} X(\omega)$$

$$+ \underbrace{\left\{\frac{1}{2} H_0(\omega) H_0(\omega - \pi) - \frac{1}{2} H_1(\omega) H_1(\omega - \pi)\right\} X(\omega - \pi)}_{2^{nd} \text{ term due to aliasing}}$$

2a

- Matrix form represpntation of Top of Page 2

$$\begin{bmatrix} V_0(\omega) \\ V_1(\omega) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} H_0(\omega) & H_0(\omega-\pi) \\ H_1(\omega) & H_1(\omega-\pi) \end{bmatrix} \begin{bmatrix} X(\omega) \\ X(\omega-\pi) \end{bmatrix}$$

- Then:

$$Y(\omega) = \begin{bmatrix} F_0(\omega) & F_1(\omega) \end{bmatrix} \begin{bmatrix} V_0(\omega) \\ V_1(\omega) \end{bmatrix}$$

• Now, substitute: $H_1(\omega) = H_0(\omega - \pi)$

$$Y(\omega) = \frac{1}{2}\left\{H_0^2(\omega) - H_0^2(\omega-\pi)\right\} X(\omega)$$
$$+ \frac{1}{2}\left\{H_0(\omega)H_0(\omega-\pi) - H_0(\omega-\pi)H_0(\omega-2\pi)\right\} X(\omega-\pi)$$

<u>Since:</u> $H_0(\omega - 2\pi) = H_0(\omega)$, by design the terms due to aliasing cancel so that the effects of aliasing are removed! $\Rightarrow$ Alias-Free

• At this point:

$$Y(\omega) = \frac{1}{2}\left\{H_0^2(\omega) - H_0^2(\omega-\pi)\right\} X(\omega)$$

• To achieve Perfect Reconstruction, require:

$$\frac{1}{2}\left\{H_0^2(\omega) - H_0^2(\omega-\pi)\right\} = e^{-jn_0\omega} \quad \text{for all } \omega$$

- Need Halfband LPF satisfying:

$$H_0^2(\omega) - H_0^2(\omega-\pi) = c\, e^{-jn_0\omega}$$

- Consider a HB LPF with a raised-cosine roll-off and a linear phase with slope $= \frac{1}{2}$

$$H_0(\omega) = H_0^{(r)}(\omega)\, e^{j\frac{\omega}{2}}$$

<u>where:</u>

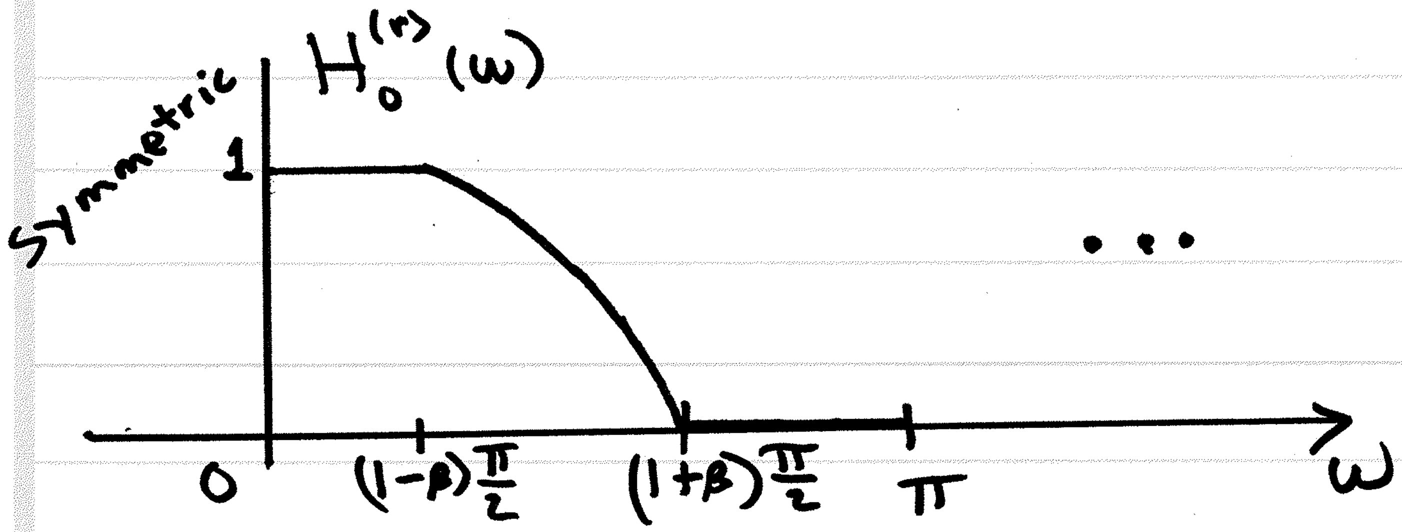
$$H_0^{(r)}(\omega) = \begin{cases} 1, & |\omega| < (1-\beta)\frac{\pi}{2} \\ \cos\left(\frac{1}{2\beta}\left[|\omega| - (1-\beta)\frac{\pi}{2}\right]\right), & (1-\beta)\frac{\pi}{2} < |\omega| < (1+\beta)\frac{\pi}{2} \\ 0, & (1+\beta)\frac{\pi}{2} < |\omega| < \pi \end{cases}$$

<u>where:</u> $0 < \beta < 1 \Rightarrow$ roll-off factor

THEN: $H_0(\omega) = H_0^{(r)}(\omega)\, e^{j\frac{\omega}{2}}$ ②

- All together:

$$H_0(\omega) = \begin{cases} e^{j\frac{\omega}{2}}, & |\omega| < (1-\beta)\frac{\pi}{2} \\ e^{j\frac{\omega}{2}} \cos\left(\frac{1}{2\beta}\left(|\omega| - (1-\beta)\frac{\pi}{2}\right)\right) & \text{for } (1-\beta)\frac{\pi}{2} < |\omega| < (1+\beta)\frac{\pi}{2} \\ 0, & (1+\beta)\frac{\pi}{2} < |\omega| < \pi \end{cases}$$



- Show that this satisfies requirement: 3

For $0<\omega<(1-\beta)\frac{\pi}{2}$:

$$H_0^2(\omega) - \underbrace{H_0^2(\omega-\pi)}_{\text{this is zero for } 0<\omega<(1-\beta)\frac{\pi}{2}}$$

THUS: $H_0^2(\omega) - H_0^2(\omega-\pi) = e^{j\frac{\omega}{2}\cdot 2} = e^{j\omega}$

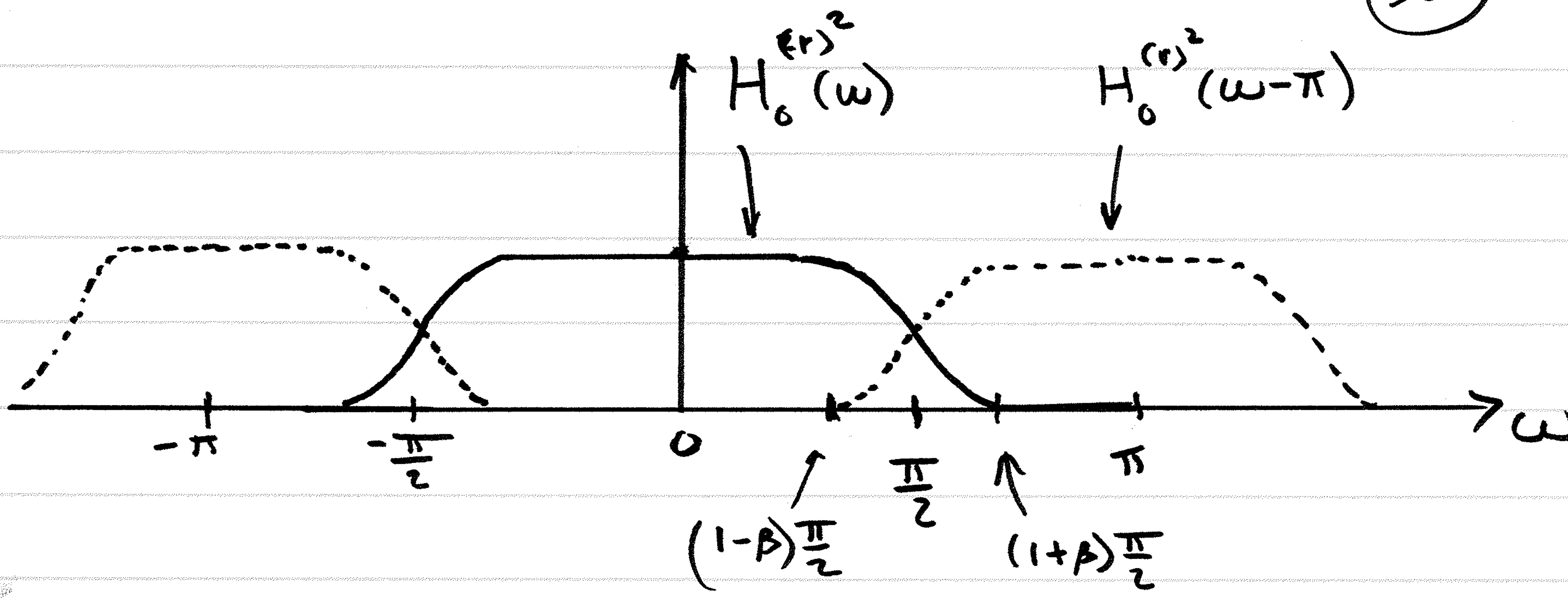
For: $(1-\beta)\frac{\pi}{2} < \omega < (1+\beta)\frac{\pi}{2}$:

$$e^{j\frac{\omega}{2}\cdot 2}\cos^2\left(\frac{1}{2\beta}\left(\omega-(1-\beta)\frac{\pi}{2}\right)\right)$$

$$-e^{j\frac{(\omega-\pi)}{2}\cdot 2}\cos^2\left(\frac{1}{2\beta}\left(|\omega-\pi|-(1-\beta)\frac{\pi}{2}\right)\right)$$

$$= e^{j\omega}\left\{\cos^2\left(\frac{1}{2\beta}\left(\omega-(1-\beta)\frac{\pi}{2}\right)\right) + \cos^2\left(\frac{1}{2\beta}\left(|\omega-\pi|-(1-\beta)\frac{\pi}{2}\right)\right)\right\}$$

3a



$$H_0^2(\omega-\pi)=0 \quad \text{for} \quad 0<\omega<(1-\beta)\frac{\pi}{2}$$

$$H_0^2(\omega)=0 \quad \text{for} \quad (1+\beta)\frac{\pi}{2}<\omega<\pi$$

$$\Rightarrow \cos^2\left(\frac{1}{2\beta}\left(\omega-(1-\beta)\frac{\pi}{2}\right)\right)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{1}{\beta}\left(\omega-(1-\beta)\frac{\pi}{2}\right)\right) \quad (4)$$

$$=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2}+\left(\frac{\omega}{\beta}-\frac{\pi}{2\beta}\right)\right)$$

$$\Rightarrow \cos^2\left(\frac{1}{2\beta}\left(|\omega-\pi|-(1-\beta)\frac{\pi}{2}\right)\right)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{1}{\beta}\left(|\omega-\pi|-(1-\beta)\frac{\pi}{2}\right)\right)$$

$$=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2}-\left(\frac{\omega}{\beta}+\frac{\pi}{2\beta}-\frac{\pi}{\beta}\right)\right)$$

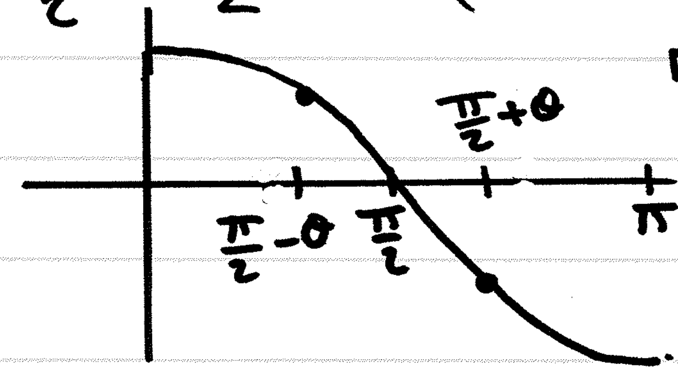
$$=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2}-\left(\frac{\omega}{\beta}-\frac{\pi}{2\beta}\right)\right)$$

where we have used the fact that:

$$|\omega-\pi|=\pi-\omega \quad \text{for} \quad (1-\beta)\frac{\pi}{2}<\omega<(1+\beta)\frac{\pi}{2}$$

· at this point, we have:

$$\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}+\theta\right) + \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}-\theta\right)$$



two terms cancel

Thus, we get the sum as 1 !

✓ checks

For: $(1+\beta)\frac{\pi}{2} < \omega < \pi$:

$$\underbrace{H_0^2(\omega)}_{\substack{\text{this term} \\ =0 \\ \text{over this} \\ \text{range}}} - H_0^2(\omega-\pi)$$

$$- e^{j\frac{(\omega-\pi)}{2}\cdot 2}$$

$$- e^{j\omega}e^{-j\pi} = e^{j\omega}$$

✓ checks

- So this works! What is the impulse response for this filter?
- It was obtained by sampling a CT pulse shape with a Square-Root Raised Cosine Spectrum

$$h_0[n] = p_{SRRC}(t)\Big|_{t=\frac{T_s}{4}+n\frac{T_s}{2}}$$

where: $p_{SRRC}(t) =$

$$= \frac{2\beta}{\pi\sqrt{T_s}} \frac{\cos\left[(1+\beta)\pi\frac{t}{T_s}\right] + \dfrac{\sin\left((1-\beta)\pi\frac{t}{T_s}\right)}{4\beta\, t/T_s}}{\left[1-\left(4\beta\frac{t}{T_s}\right)^2\right]}$$

OR: $h_0[n] = p_{SRRC}\left(t+\frac{T_s}{4}\right)\Big|_{t=n\frac{T_s}{2}}$

- sampling every $\frac{T_s}{2}$ secs, then $\frac{T_s}{4}$ corresponds to a shift of a half-sample $\Rightarrow$ this is what gives rise to the linear phase term $e^{j\frac{\omega}{2}}$

$$P_{SRRC}(f) = \alpha \begin{cases} 1, & 0 \leq |f| \leq \frac{1-\beta}{2T_s} \\ \cos\left[\frac{\pi T_s}{2\beta}\left(|f| - \frac{1-\beta}{2T_s}\right)\right], & \text{for } \frac{1-\beta}{2T_s} \leq |f| \leq \frac{1+\beta}{2T_s} \\ 0, & |f| > \frac{1+\beta}{2T_s} \end{cases}$$

Since $T_s > 0$:

$$P_{SRRC}(f) = \alpha \cos\left[\frac{\pi}{2\beta}\left(|fT_s| - \frac{1-\beta}{2}\right)\right]$$

$$= \alpha \cos\left[\frac{1}{2\beta}\left(|\pi f T_s| - (1-\beta)\frac{\pi}{2}\right)\right]$$

This simple two-tap filter pair also works

$$h_0[n] = \{\underset{\uparrow}{1}, 1\} \qquad h_1[n] = \{\underset{\uparrow}{1}, -1\} = (-1)^n h_0[n]$$

$$H_0(\omega) = 1 + e^{-j\omega} \qquad H_1(\omega) = 1 - e^{-j\omega}$$

$$= 2e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right) \qquad = 2j\, e^{-j\frac{\omega}{2}} \sin\left(\frac{\omega}{2}\right)$$

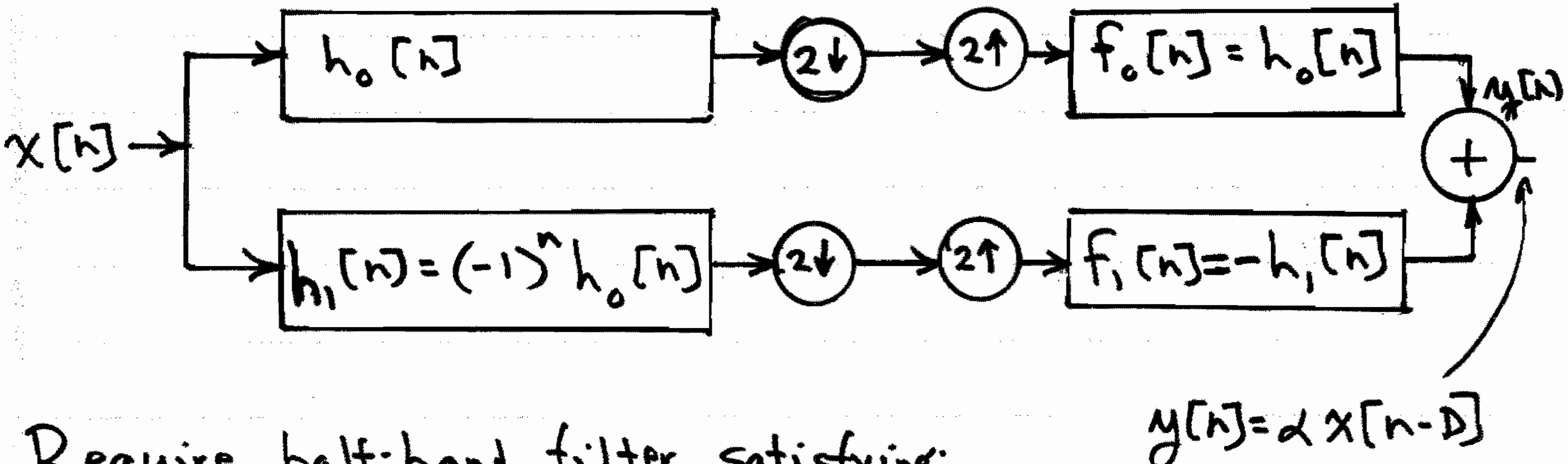
$$H_0^2(\omega) - H_1^2(\omega) = H_0^2(\omega) - H_0^2(\omega - \pi)$$

$$4e^{-j\omega}\left\{\cos^2\left(\frac{\omega}{2}\right) - (-1)\sin^2\left(\frac{\omega}{2}\right)\right\}$$

$$= 4e^{-j\omega}$$

THUS: $F_0(\omega) = \dfrac{H_0(\omega)\, 4e^{-j\omega}}{4e^{-j\omega}} = H_0(\omega)$

Summarizing the Two-Channel QMF:

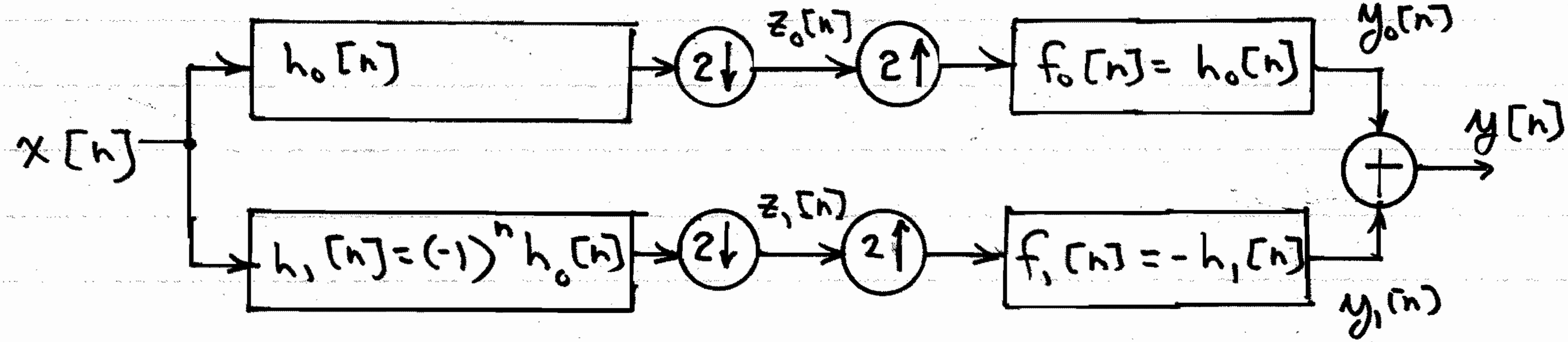


Require half-band filter satisfying:

$$H_0^2(\omega) - H_0^2(\omega - \pi) = c\,e^{-M\omega} \quad \text{for all } \omega$$

$$Z_0(\omega) = \frac{1}{2} H_0\left(\frac{\omega}{2}\right) X\left(\frac{\omega}{2}\right) + \frac{1}{2} H_0\left(\frac{\omega - 2\pi}{2}\right) X\left(\frac{\omega - 2\pi}{2}\right)$$

$$Y_0(\omega) = \frac{1}{2} H_0^2(\omega) X(\omega) + \frac{1}{2} H_0(\omega) H_0(\omega - \pi) X(\omega - \pi)$$



$$Z_1(\omega) = \frac{1}{2} H_1\left(\frac{\omega}{2}\right) X\left(\frac{\omega}{2}\right) + \frac{1}{2} H_1\left(\frac{\omega - 2\pi}{2}\right) X\left(\frac{\omega - 2\pi}{2}\right)$$

$$Y_1(\omega) = -H_0^2(\omega - \pi) \frac{1}{2} X(\omega) - H_0(\omega) H_0(\omega - \pi) \frac{1}{2} X(\omega - \pi)$$

$$H_1(\omega) = H_0(\omega - \pi)$$

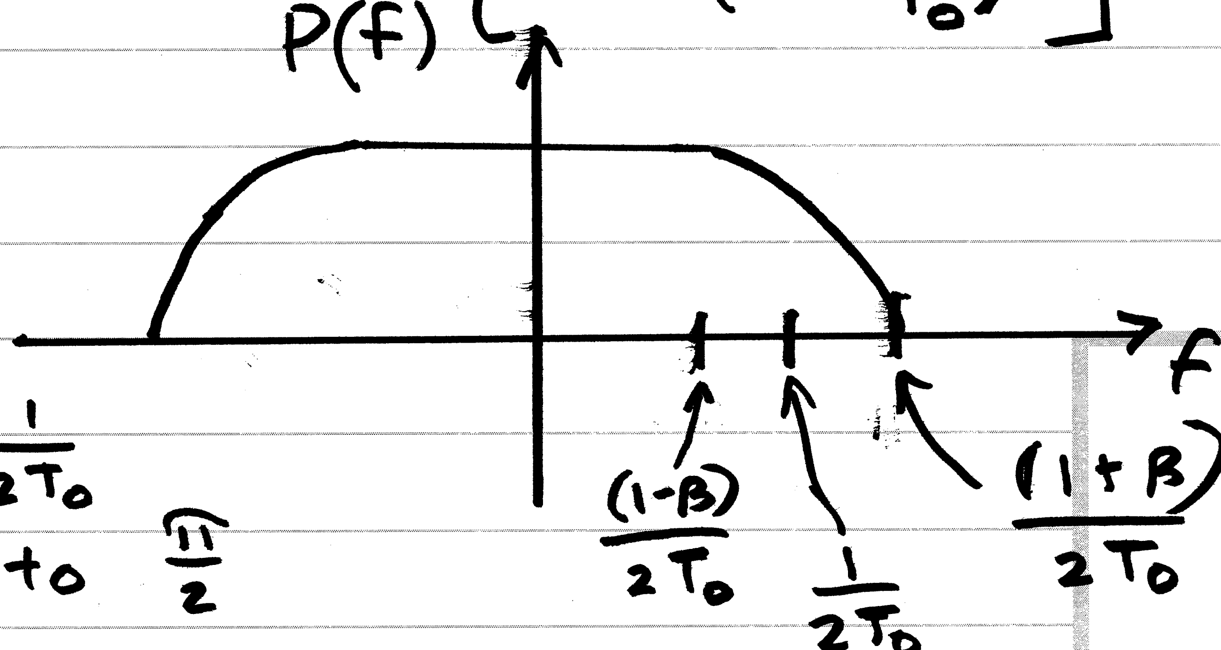
$$\boxed{Y(\omega) = \left\{ H_0^2(\omega) - H_0^2(\omega - \pi) \right\} \frac{1}{2} X(\omega)}$$

H_1 (w - pi) = H_0 (w - 2 pi) = H_0 (w)

- Cosine Roll-Off Filter / Pulse-Shape

- <u>analog case:</u>

$$p(t) = \frac{2\beta}{\pi\sqrt{T_0}} \; \frac{\cos\left[(1+\beta)\pi \frac{t}{T_0}\right] + \dfrac{\sin\left((1-\beta)\pi \frac{t}{T_0}\right)}{4\beta t / T_0}}{\left[1 - \left(4\beta \frac{t}{T_0}\right)^2\right]}$$



- Desire $\frac{1}{2T_0}$ mapped to $\frac{\pi}{2}$

$$\omega = 2\pi \frac{f}{F_s} = 2\pi f T_s$$

$$\Rightarrow \frac{\pi}{2} = 2\pi \frac{1}{2T_0} T_s \Rightarrow T_s = \frac{T_0}{2}$$

- Yields DT Halfband Filter:

$$\hat{h}[n] = \left\{ \frac{2\beta \cos\left((1+\beta)\frac{\pi}{2}n\right)}{\pi\left(1-4\beta^2 n^2\right)} + \frac{\sin\left((1-\beta)\frac{\pi}{2}n\right)}{\pi\left(n-4\beta^2 n^3\right)} \right\} \sqrt{2}$$

- To get linear phase $e^{j\frac{\omega}{2}}$, replace n by $n+.5$

$$h_{HB}[n] = \left\{ \frac{2\beta \cos\left((1+\beta)\frac{\pi}{2}(n+.5)\right)}{\pi\left(1-4\beta^2 (n+.5)^2\right)} + \frac{\sin\left((1-\beta)\frac{\pi}{2}(n+.5)\right)}{\pi\left(n-4\beta^2 (n+.5)^3\right)} \right\} \sqrt{2}$$

$(n+.5)$

- See Matlab code:

PRRC2chan.m