CHAPTER

3

The z-Transform and Its Application to the Analysis of LTI Systems

Transform techniques are an important tool in the analysis of signals and linear time-invariant (LTI) systems. In this chapter we introduce the z-transform, develop its properties, and demonstrate its importance in the analysis and characterization of linear time-invariant systems.

The z-transform plays the same role in the analysis of discrete-time signals and LTI systems as the Laplace transform does in the analysis of continuous-time signals and LTI systems. For example, we shall see that in the z-domain (complex z-plane) the convolution of two time-domain signals is equivalent to multiplication of their corresponding z-transforms. This property greatly simplifies the analysis of the response of an LTI system to various signals. In addition, the z-transform provides us with a means of characterizing an LTI system, and its response to various signals, by its pole–zero locations.

We begin this chapter by defining the z-transform. Its important properties are presented in Section 3.2. In Section 3.3 the transform is used to characterize signals in terms of their pole–zero patterns. Section 3.4 describes methods for inverting the z-transform of a signal so as to obtain the time-domain representation of the signal. Finally, in Section 3.6, we treat one-sided z-transform and use it to solve linear difference equations with nonzero initial conditions. Section 3.5 is focused on the use of the z-transform in the analysis of LTI systems.

3.1 The z-Transform

In this section we introduce the z-transform of a discrete-time signal, investigate its convergence properties, and briefly discuss the inverse z-transform.

3.1.1 The Direct z-Transform

The z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{3.1.1}$$

where z is a complex variable. The relation (3.1.1) is sometimes called the *direct z-transform* because it transforms the time-domain signal $x(n)$ into its complex-plane representation $X(z)$. The inverse procedure [i.e., obtaining $x(n)$ from $X(z)$] is called the *inverse z-transform* and is examined briefly in Section 3.1.2 and in more detail in Section 3.4.

For convenience, the z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv Z\{x(n)\} \tag{3.1.2}$$

whereas the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \overset{z}{\longleftrightarrow} X(z) \tag{3.1.3}$$

Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges. The *region of convergence* (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value. Thus any time we cite a z-transform we should also indicate its ROC.

We illustrate these concepts by some simple examples.

EXAMPLE 3.1.1

Determine the z-transforms of the following *finite-duration* signals.

(a) $x_1(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

(b) $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

(c) $x_3(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$

(d) $x_4(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

(e) $x_5(n) = \delta(n)$

(f) $x_6(n) = \delta(n-k), k > 0$

(g) $x_7(n) = \delta(n+k), k > 0$

Solution. From definition (3.1.1), we have

(a) $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$, ROC: entire z-plane except $z = 0$

(b) $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane except $z = 0$ and $z = \infty$

(c) $X_3(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$, ROC: entire z-plane except $z = 0$

(d) $X_4(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane except $z = 0$ and $z = \infty$

(e) $X_5(z) = 1$ [i.e., $\delta(n) \overset{z}{\longleftrightarrow} 1$], ROC: entire z-plane

(f) $X_6(z) = z^{-k}$ [i.e., $\delta(n-k) \overset{z}{\longleftrightarrow} z^{-k}$], $k > 0$, ROC: entire z-plane except $z = 0$

(g) $X_7(z) = z^k$ [i.e., $\delta(n+k) \overset{z}{\longleftrightarrow} z^k$], $k > 0$, ROC: entire z-plane except $z = \infty$

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From this example it is easily seen that the ROC of a *finite-duration signal* is the entire z-plane, except possibly the points $z = 0$ and/or $z = \infty$. These points are excluded, because z^k $(k > 0)$ becomes unbounded for $z = \infty$ and z^{-k} $(k > 0)$ becomes unbounded for $z = 0$.

From a mathematical point of view the z-transform is simply an alternative representation of a signal. This is nicely illustrated in Example 3.1.1, where we see that the coefficient of z^{-n}, in a given transform, is the value of the signal at time n. In other words, the exponent of z contains the time information we need to identify the samples of the signal.

In many cases we can express the sum of the finite or infinite series for the z-transform in a closed-form expression. In such cases the z-transform offers a compact alternative representation of the signal.

EXAMPLE 3.1.2

Determine the z-transform of the signal

$$x(n) = (\frac{1}{2})^n u(n)$$

Solution. The signal $x(n)$ consists of an infinite number of nonzero values

$$x(n) = \{1, (\frac{1}{2}), (\frac{1}{2})^2, (\frac{1}{2})^3, \ldots, (\frac{1}{2})^n, \ldots\}$$

The z-transform of $x(n)$ is the infinite power series

$$X(z) = 1 + \frac{1}{2}z^{-1} + (\frac{1}{2})^2 z^{-2} + (\frac{1}{2})^n z^{-n} + \cdots$$

$$= \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} = \sum_{n=0}^{\infty} (\frac{1}{2}z^{-1})^n$$

This is an infinite geometric series. We recall that

$$1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A} \qquad \text{if } |A| < 1$$

Consequently, for $|\frac{1}{2}z^{-1}| < 1$, or equivalently, for $|z| > \frac{1}{2}$, $X(z)$ converges to

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad \text{ROC: } |z| > \frac{1}{2}$$

We see that in this case, the z-transform provides a compact alternative representation of the signal $x(n)$.

Let us express the complex variable z in polar form as

$$z = re^{j\theta} \tag{3.1.4}$$

where $r = |z|$ and $\theta = \measuredangle z$. Then $X(z)$ can be expressed as

$$X(z)|_{z=re^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\theta n}$$

In the ROC of $X(z)$, $|X(z)| < \infty$. But

$$\begin{aligned} |X(z)| &= \left| \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\theta n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}e^{-j\theta n}| = \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| \end{aligned} \tag{3.1.5}$$

Hence $|X(z)|$ is finite if the sequence $x(n)r^{-n}$ is absolutely summable.

The problem of finding the ROC for $X(z)$ is equivalent to determining the range of values of r for which the sequence $x(n)r^{-n}$ is absolutely summable. To elaborate, let us express (3.1.5) as

$$\begin{aligned} |X(z)| &\leq \sum_{n=-\infty}^{-1} |x(n)r^{-n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| \\ &\leq \sum_{n=1}^{\infty} |x(-n)r^n| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| \end{aligned} \tag{3.1.6}$$

If $X(z)$ converges in some region of the complex plane, both summations in (3.1.6) must be finite in that region. If the first sum in (3.1.6) converges, there must exist values of r small enough such that the product sequence $x(-n)r^n$, $1 \leq n < \infty$, is absolutely summable. Therefore, the ROC for the first sum consists of all points in a circle of some radius r_1, where $r_1 < \infty$, as illustrated in Fig. 3.1.1(a). On the other hand, if the second sum in (3.1.6) converges, there must exist values of r large enough such that the product sequence $x(n)/r^n$, $0 \leq n < \infty$, is absolutely summable. Hence the ROC for the second sum in (3.1.6) consists of all points outside a circle of radius $r > r_2$, as illustrated in Fig. 3.1.1(b).

Since the convergence of $X(z)$ requires that both sums in (3.1.6) be finite, it follows that the ROC of $X(z)$ is generally specified as the annular region in the z-plane, $r_2 < r < r_1$, which is the common region where both sums are finite. This region is illustrated in Fig. 3.1.1(c). On the other hand, if $r_2 > r_1$, there is no common region of convergence for the two sums and hence $X(z)$ does not exist.

The following examples illustrate these important concepts.

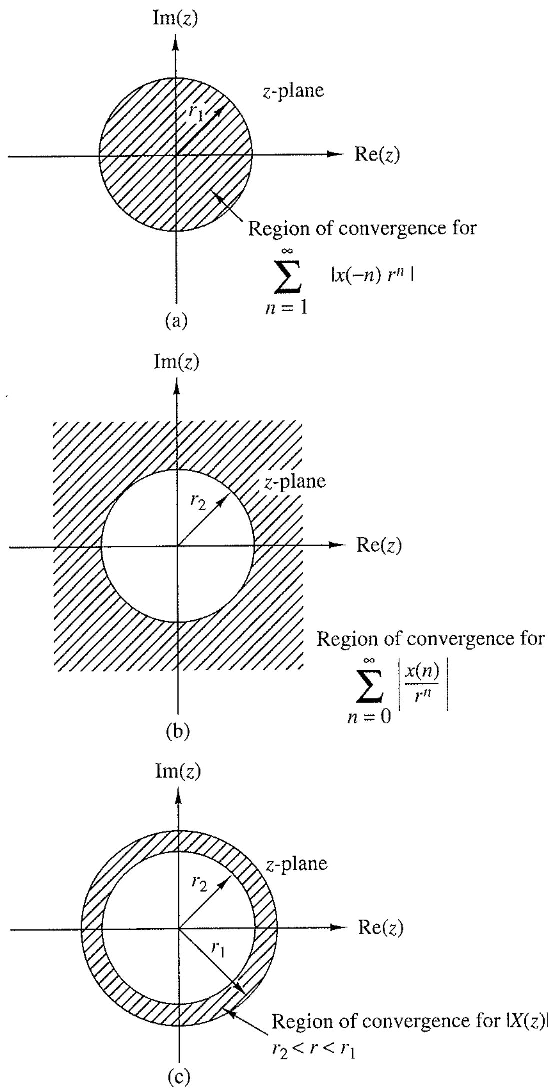


Figure 3.1.1
Region of convergence for $X(z)$ and its corresponding causal and anticausal components.

EXAMPLE 3.1.3

Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

Solution. From the definition (3.1.1) we have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

If $|\alpha z^{-1}| < 1$ or equivalently, $|z| > |\alpha|$, this power series converges to $1/(1-\alpha z^{-1})$. Thus we have the z-transform pair

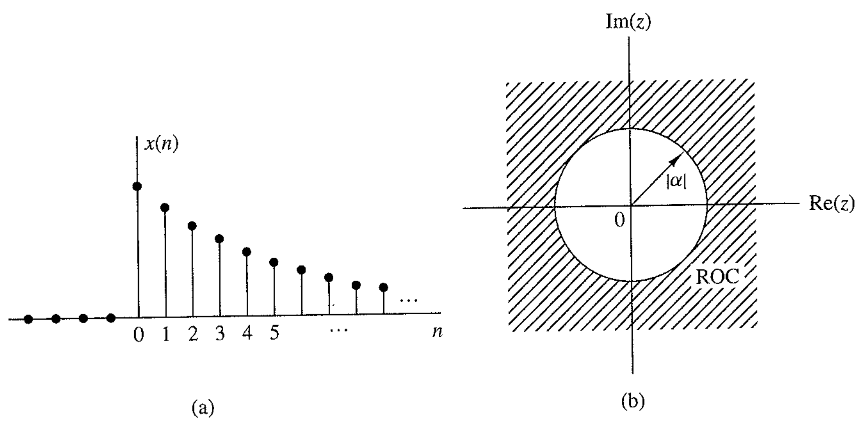


Figure 3.1.2 The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z-transform (b).

$$x(n) = \alpha^n u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1-\alpha z^{-1}}, \qquad \text{ROC: } |z| > |\alpha| \tag{3.1.7}$$

The ROC is the exterior of a circle having radius $|\alpha|$. Figure 3.1.2 shows a graph of the signal $x(n)$ and its corresponding ROC. Note that, in general, α need not be real.

If we set $\alpha = 1$ in (3.1.7), we obtain the z-transform of the unit step signal

$$x(n) = u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1-z^{-1}}, \qquad \text{ROC: } |z| > 1 \tag{3.1.8}$$

EXAMPLE 3.1.4

Determine the z-transform of the signal

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

Solution. From the definition (3.1.1) we have

$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n} = -\sum_{l=1}^{\infty} (\alpha^{-1} z)^l$$

where $l = -n$. Using the formula

$$A + A^2 + A^3 + \cdots = A(1 + A + A^2 + \cdots) = \frac{A}{1-A}$$

when $|A| < 1$ gives

$$X(z) = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z} = \frac{1}{1-\alpha z^{-1}}$$

provided that $|\alpha^{-1} z| < 1$ or, equivalently, $|z| < |\alpha|$. Thus

$$x(n) = -\alpha^n u(-n-1) \overset{z}{\longleftrightarrow} X(z) = -\frac{1}{1-\alpha z^{-1}}, \qquad \text{ROC: } |z| < |\alpha| \tag{3.1.9}$$

The ROC is now the interior of a circle having radius $|\alpha|$. This is shown in Fig. 3.1.3.

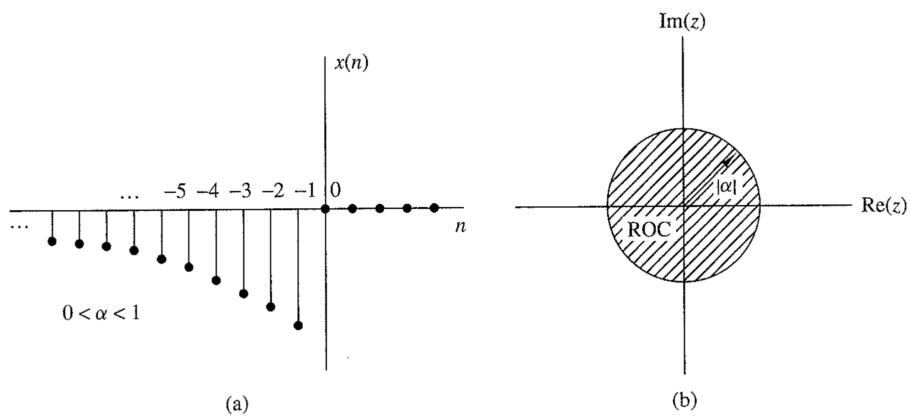


Figure 3.1.3 Anticausal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).

Examples 3.1.3 and 3.1.4 illustrate two very important issues. The first concerns the uniqueness of the z-transform. From (3.1.7) and (3.1.9) we see that the causal signal $\alpha^n u(n)$ and the anticausal signal $-\alpha^n u(-n-1)$ have identical closed-form expressions for the z-transform, that is,

$$Z\{\alpha^n u(n)\} = Z\{-\alpha n u(-n-1)\} = \frac{1}{1-\alpha z^{-1}}$$

This implies that a closed-form expression for the z-transform does not uniquely specify the signal in the time domain. The ambiguity can be resolved only if in addition to the closed-form expression, the ROC is specified. In summary, *a discrete-time signal $x(n)$ is uniquely determined by its z-transform $X(z)$ and the region of convergence of $X(z)$.* In this text the term "z-transform" is used to refer to both the closed-form expression and the corresponding ROC. Example 3.1.3 also illustrates the point that *the ROC of a causal signal is the exterior of a circle of some radius r_2 while the ROC of an anticausal signal is the interior of a circle of some radius r_1.* The following example considers a sequence that is nonzero for $-\infty < n < \infty$.

EXAMPLE 3.1.5

Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

Solution. From definition (3.1.1) we have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{l=1}^{\infty} (b^{-1}z)^l$$

The first power series converges if $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$. The second power series converges if $|b^{-1}z| < 1$ or $|z| < |b|$.

In determining the convergence of $X(z)$, we consider two different cases.

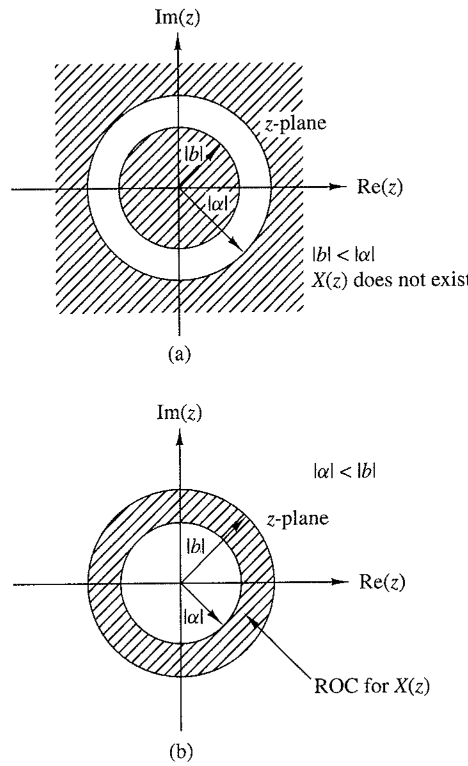


Figure 3.1.4
ROC for z-transform in Example 3.1.5.

Case 1 $|b| < |\alpha|$: In this case the two ROC above do not overlap, as shown in Fig. 3.1.4(a). Consequently, we cannot find values of z for which both power series converge simultaneously. Clearly, in this case, $X(z)$ does not exist.

Case 2 $|b| > |\alpha|$: In this case there is a ring in the z-plane where both power series converge simultaneously, as shown in Fig. 3.1.4(b). Then we obtain

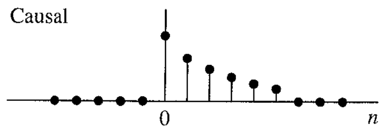
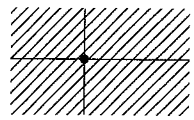
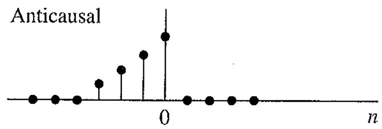
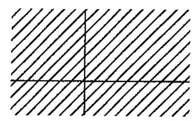
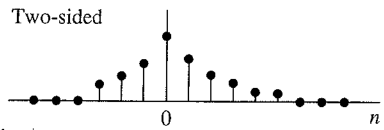
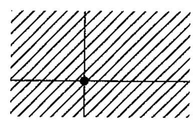
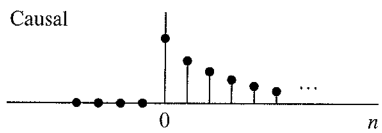
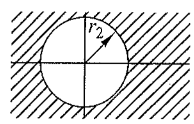
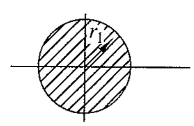
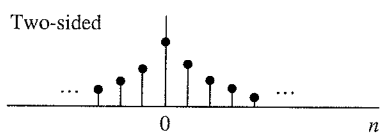
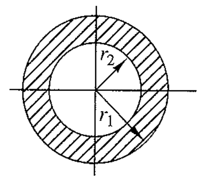
$$\begin{aligned} X(z) &= \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}} \\ &= \frac{b-\alpha}{\alpha + b - z - \alpha b z^{-1}} \end{aligned} \tag{3.1.10}$$

The ROC of $X(z)$ is $|\alpha| < |z| < |b|$.

This example shows that *if there is a ROC for an infinite-duration two-sided signal, it is a ring (annular region) in the z-plane.* From Examples 3.1.1, 3.1.3, 3.1.4, and 3.1.5, we see that the ROC of a signal depends both on its duration (finite or infinite) and on whether it is causal, anticausal, or two-sided. These facts are summarized in Table 3.1.

One special case of a two-sided signal is a signal that has infinite duration on the right side but not on the left [i.e., $x(n) = 0$ for $n < n_0 < 0$]. A second case is

TABLE 3.1 Characteristic Families of Signals with Their Corresponding ROCs

	Signal	ROC	
Finite-Duration Signals			
Causal			Entire z-plane except $z = 0$
Anticausal			Entire z-plane except $z = \infty$
Two-sided			Entire z-plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals			
Causal			$\|z\| > r_2$
Anticausal			$\|z\| < r_1$
Two-sided			$r_2 < \|z\| < r_1$

a signal that has infinite duration on the left side but not on the right [i.e., $x(n) = 0$ for $n > n_1 > 0$]. A third special case is a signal that has finite duration on both the left and right sides [i.e., $x(n) = 0$ for $n < n_0 < 0$ and $n > n_1 > 0$]. These types of signals are sometimes called *right-sided, left-sided,* and *finite-duration two-sided* signals, respectively. The determination of the ROC for these three types of signals is left as an exercise for the reader (Problem 3.5).

Finally, we note that the z-transform defined by (3.1.1) is sometimes referred to as the *two-sided* or *bilateral z-transform*, to distinguish it from the *one-sided* or *unilateral z-transform* given by

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{3.1.11}$$

The one-sided z-transform is examined in Section 3.6. In this text we use the expression z-transform exclusively to mean the two-sided z-transform defined by (3.1.1). The term "two-sided" will be used only in cases where we want to resolve any ambiguities. Clearly, if $x(n)$ is causal [i.e., $x(n) = 0$ for $n < 0$], the one-sided and two-sided z-transforms are identical. In any other case, they are different.

3.1.2 The Inverse z-Transform

Often, we have the z-transform $X(z)$ of a signal and we must determine the signal sequence. The procedure for transforming from the z-domain to the time domain is called the *inverse z-transform*. An inversion formula for obtaining $x(n)$ from $X(z)$ can be derived by using the *Cauchy integral theorem*, which is an important theorem in the theory of complex variables.

To begin, we have the z-transform defined by (3.1.1) as

$$X(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \tag{3.1.12}$$

Suppose that we multiply both sides of (3.1.12) by z^{n-1} and integrate both sides over a closed contour within the ROC of $X(z)$ which encloses the origin. Such a contour is illustrated in Fig. 3.1.5. Thus we have

$$\oint_C X(z)z^{n-1}\,dz = \oint_C \sum_{k=-\infty}^{\infty} x(k)z^{n-1-k}\,dz \tag{3.1.13}$$

where C denotes the closed contour in the ROC of $X(z)$, taken in a counterclockwise direction. Since the series converges on this contour, we can interchange the order of

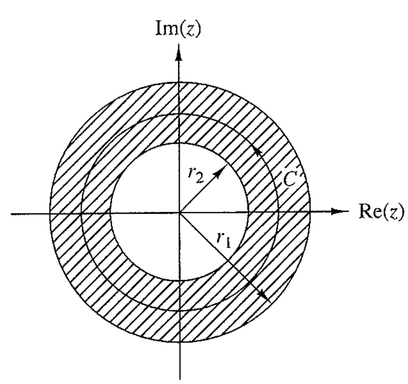


Figure 3.1.5
Contour C for integral in (3.1.13).

integration and summation on the right-hand side of (3.1.13). Thus (3.1.13) becomes

$$\oint_C X(z)z^{n-1}dz = \sum_{k=-\infty}^{\infty} x(k) \oint_C z^{n-1-k}dz \tag{3.1.14}$$

Now we can invoke the Cauchy integral theorem, which states that

$$\frac{1}{2\pi j} \oint_C z^{n-1-k}\, dz = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} \tag{3.1.15}$$

where C is any contour that encloses the origin. By applying (3.1.15), the right-hand side of (3.1.14) reduces to $2\pi j x(n)$ and hence the desired inversion formula

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}\, dz \tag{3.1.16}$$

Although the contour integral in (3.1.16) provides the desired inversion formula for determining the sequence $x(n)$ from the z-transform, we shall not use (3.1.16) directly in our evaluation of inverse z-transforms. In our treatment we deal with signals and systems in the z-domain which have rational z-transforms (i.e., z-transforms that are a ratio of two polynomials). For such z-transforms we develop a simpler method for inversion that stems from (3.1.16) and employs a table lookup.

3.2 Properties of the z-Transform

The z-transform is a very powerful tool for the study of discrete-time signals and systems. The power of this transform is a consequence of some very important properties that the transform possesses. In this section we examine some of these properties.

In the treatment that follows, it should be remembered that when we combine several z-transforms, the ROC of the overall transform is, at least, the intersection of the ROC of the individual transforms. This will become more apparent later, when we discuss specific examples.

Linearity. If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$

and

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = a_1x_1(n) + a_2x_2(n) \overset{z}{\longleftrightarrow} X(z) = a_1X_1(z) + a_2X_2(z) \tag{3.2.1}$$

for any constants a_1 and a_2. The proof of this property follows immediately from the definition of linearity and is left as an exercise for the reader.

The linearity property can easily be generalized for an arbitrary number of signals. Basically, it implies that the z-transform of a linear combination of signals is the same linear combination of their z-transforms. Thus the linearity property helps us to find the z-transform of a signal by expressing the signal as a sum of elementary signals, for each of which, the z-transform is already known.

EXAMPLE 3.2.1

Determine the z-transform and the ROC of the signal

$$x(n) = [3(2^n) - 4(3^n)]u(n)$$

Solution. If we define the signals

$$x_1(n) = 2^n u(n)$$

and

$$x_2(n) = 3^n u(n)$$

then $x(n)$ can be written as

$$x(n) = 3x_1(n) - 4x_2(n)$$

According to (3.2.1), its z-transform is

$$X(z) = 3X_1(z) - 4X_2(z)$$

From (3.1.7) we recall that

$$\alpha^n u(n) \stackrel{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}, \qquad \text{ROC: } |z| > |\alpha| \tag{3.2.2}$$

By setting $\alpha = 2$ and $\alpha = 3$ in (3.2.2), we obtain

$$x_1(n) = 2^n u(n) \stackrel{z}{\longleftrightarrow} X_1(z) = \frac{1}{1-2z^{-1}}, \qquad \text{ROC: } |z| > 2$$

$$x_2(n) = 3^n u(n) \stackrel{z}{\longleftrightarrow} X_2(z) = \frac{1}{1-3z^{-1}}, \qquad \text{ROC: } |z| > 3$$

The intersection of the ROC of $X_1(z)$ and $X_2(z)$ is $|z| > 3$. Thus the overall transform $X(z)$ is

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}, \qquad \text{ROC: } |z| > 3$$

EXAMPLE 3.2.2

Determine the z-transform of the signals

(a) $x(n) = (\cos \omega_0 n)u(n)$

(b) $x(n) = (\sin \omega_0 n)u(n)$

Solution.

(a) By using Euler's identity, the signal $x(n)$ can be expressed as

$$x(n) = (\cos \omega_0 n)u(n) = \frac{1}{2}e^{j\omega_0 n}u(n) + \frac{1}{2}e^{-j\omega_0 n}u(n)$$

Thus (3.2.1) implies that

$$X(z) = \frac{1}{2}Z\{e^{j\omega_0 n}u(n)\} + \frac{1}{2}Z\{e^{-j\omega_0 n}u(n)\}$$

If we set $\alpha = e^{\pm j\omega_0}(|\alpha| = |e^{\pm j\omega_0}| = 1)$ in (3.2.2), we obtain

$$e^{j\omega_0 n}u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - e^{j\omega_0}z^{-1}}, \qquad \text{ROC: } |z| > 1$$

and

$$e^{-j\omega_0 n}u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - e^{-j\omega_0}z^{-1}}, \qquad \text{ROC: } |z| > 1$$

Thus

$$X(z) = \frac{1}{2}\frac{1}{1 - e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-j\omega_0}z^{-1}}, \qquad \text{ROC: } |z| > 1$$

After some simple algebraic manipulations we obtain the desired result, namely,

$$(\cos\omega_0 n)u(n) \overset{z}{\longleftrightarrow} \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}, \qquad \text{ROC: } |z| > 1 \tag{3.2.3}$$

(b) From Euler's identity,

$$x(n) = (\sin\omega 0 n)u(n) = \frac{1}{2j}[e^{j\omega_0 n}u(n) - e^{-j\omega_0 n}u(n)]$$

Thus

$$X(z) = \frac{1}{2j}\left(\frac{1}{1 - e^{j\omega_0}z^{-1}} - \frac{1}{1 - e^{-j\omega_0}z^{-1}}\right), \qquad \text{ROC: } |z| > 1$$

and finally,

$$(\sin\omega_0 n)u(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}\sin\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}, \qquad \text{ROC: } |z| > 1 \tag{3.2.4}$$

Time shifting. If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k}X(z) \tag{3.2.5}$$

The ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$. The proof of this property follows immediately from the definition of the z-transform given in (3.1.1)

The properties of linearity and time shifting are the key features that make the z-transform extremely useful for the analysis of discrete-time LTI systems.

EXAMPLE 3.2.3

By applying the time-shifting property, determine the z-transform of the signals $x_2(n)$ and $x_3(n)$ in Example 3.1.1 from the z-transform of $x_1(n)$.

Solution. It can easily be seen that

$$x_2(n) = x_1(n+2)$$

and

$$x_3(n) = x_1(n-2)$$

Thus from (3.2.5) we obtain

$$X_2(z) = z^2 X_1(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$$

and

$$X_3(z) = z^{-2} X_1(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

Note that because of the multiplication by z^2, the ROC of $X_2(z)$ does not include the point $z = \infty$, even if it is contained in the ROC of $X_1(z)$.

Example 3.2.3 provides additional insight in understanding the meaning of the shifting property. Indeed, if we recall that the coefficient of z^{-n} is the sample value at time n, it is immediately seen that delaying a signal by $k (k > 0)$ samples [i.e., $x(n) \to x(n-k)$] corresponds to multiplying all terms of the z-transform by z^{-k}. The coefficient of z^{-n} becomes the coefficient of $z^{-(n+k)}$.

EXAMPLE 3.2.4

Determine the transform of the signal

$$x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases} \tag{3.2.6}$$

Solution. We can determine the z-transform of this signal by using the definition (3.1.1). Indeed,

$$X(z) = \sum_{n=0}^{N-1} 1 \cdot z^{-n} = 1 + z^{-1} + \cdots + z^{-(N-1)} = \begin{cases} N, & \text{if } z = 1 \\ \dfrac{1 - z^{-N}}{1 - z^{-1}}, & \text{if } z \neq 1 \end{cases} \tag{3.2.7}$$

Since $x(n)$ has finite duration, its ROC is the entire z-plane, except $z = 0$.

Let us also derive this transform by using the linearity and time-shifting properties. Note that $x(n)$ can be expressed in terms of two unit step signals

$$x(n) = u(n) - u(n-N)$$

By using (3.2.1) and (3.2.5) we have

$$X(z) = Z\{u(n)\} - Z\{u(n-N)\} = (1 - z^{-N}) Z\{u(n)\} \tag{3.2.8}$$

However, from (3.1.8) we have

$$Z\{u(n)\} = \frac{1}{1 - z^{-1}}, \qquad \text{ROC: } |z| > 1$$

which, when combined with (3.2.8), leads to (3.2.7).

Example 3.2.4 helps to clarify a very important issue regarding the ROC of the combination of several z-transforms. If the linear combination of several signals has finite duration, the ROC of its z-transform is exclusively dictated by the finite-duration nature of this signal, not by the ROC of the individual transforms.

Scaling in the z-domain. If

$$x(n) \overset{z}{\longleftrightarrow} X(z), \qquad \text{ROC: } r_1 < |z| < r_2$$

then

$$a^n x(n) \overset{z}{\longleftrightarrow} X(a^{-1}z), \qquad \text{ROC: } |a|r_1 < |z| < |a|r_2 \tag{3.2.9}$$

for any constant a, real or complex.

Proof From the definition (3.1.1)

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n}$$

$$= X(a^{-1}z)$$

Since the ROC of $X(z)$ is $r_1 < |z| < r_2$, the ROC of $X(a^{-1}z)$ is

$$r_1 < |a^{-1}z| < r_2$$

or

$$|a|r_1 < |z| < |a|r_2$$

To better understand the meaning and implications of the scaling property, we express a and z in polar form as $a = r_0 e^{j\omega_0}$, $z = re^{j\omega}$, and we introduce a new complex variable $w = a^{-1}z$. Thus $Z\{x(n)\} = X(z)$ and $Z\{a^n x(n)\} = X(w)$. It can easily be seen that

$$w = a^{-1}z = \left(\frac{1}{r_0} r\right) e^{j(\omega - \omega_0)}$$

This change of variables results in either shrinking (if $r_0 > 1$) or expanding (if $r_0 < 1$) the z-plane in combination with a rotation (if $\omega_0 \neq 2k\pi$) of the z-plane (see Fig. 3.2.1). This explains why we have a change in the ROC of the new transform where $|a| < 1$. The case $|a| = 1$, that is, $a = e^{j\omega_0}$ is of special interest because it corresponds only to rotation of the z-plane.

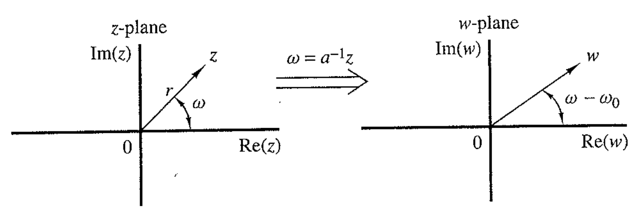


Figure 3.2.1 Mapping of the z-plane to the w-plane via the transformation $\omega = a^{-1}z$, $a = r_0 e^{j\omega_0}$.

EXAMPLE 3.2.5

Determine the z-transforms of the signals

(a) $x(n) = a^n(\cos\omega_0 n)u(n)$

(b) $x(n) = a^n(\sin\omega_0 n)u(n)$

Solution.

(a) From (3.2.3) and (3.2.9) we easily obtain

$$a^n(\cos\omega_0 n)u(n) \overset{z}{\longleftrightarrow} \frac{1 - az^{-1}\cos\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2z^{-2}}, \qquad |z| > |a| \tag{3.2.10}$$

(b) Similarly, (3.2.4) and (3.2.9) yield

$$a^n(\sin\omega_0 n)u(n) \overset{z}{\longleftrightarrow} \frac{az^{-1}\sin\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2z^{-2}}, \qquad |z| > |a| \tag{3.2.11}$$

Time reversal. If

$$x(n) \overset{z}{\longleftrightarrow} X(z), \qquad \text{ROC: } r_1 < |z| < r_2$$

then

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}), \qquad \text{ROC: } \frac{1}{r_2} < |z| < \frac{1}{r_1} \tag{3.2.12}$$

Proof From the definition (3.1.1), we have

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)z^{-n} = \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l} = X(z^{-1})$$

where the change of variable $l = -n$ is made. The ROC of $X(z^{-1})$ is

$$r_1 < |z^{-1}| < r_2 \quad \text{or equivalently} \quad \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Note that the ROC for $x(n)$ is the inverse of that for $x(-n)$. This means that if z_0 belongs to the ROC of $x(n)$, then $1/z_0$ is in the ROC for $x(-n)$.

An intuitive proof of (3.2.12) is the following. When we fold a signal, the coefficient of z^{-n} becomes the coefficient of z^n. Thus, folding a signal is equivalent to replacing z by z^{-1} in the z-transform formula. In other words, reflection in the time domain corresponds to inversion in the z-domain.

EXAMPLE 3.2.6

Determine the z-transform of the signal

$$x(n) = u(-n)$$

Solution. It is known from (3.1.8) that

$$u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \qquad \text{ROC: } |z| > 1$$

By using (3.2.12), we easily obtain

$$u(-n) \overset{z}{\longleftrightarrow} \frac{1}{1 - z}, \qquad \text{ROC: } |z| < 1 \tag{3.2.13}$$

Differentiation in the z-domain. If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} \tag{3.2.14}$$

Proof By differentiating both sides of (3.1.1), we have

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)(-n)z^{-n-1} = -z^{-1} \sum_{n=-\infty}^{\infty} [nx(n)]z^{-n}$$

$$= -z^{-1} Z\{nx(n)\}$$

Note that both transforms have the same ROC.

EXAMPLE 3.2.7

Determine the z-transform of the signal

$$x(n) = na^n u(n)$$

Solution. The signal $x(n)$ can be expressed as $nx_1(n)$, where $x_1(n) = a^n u(n)$. From (3.2.2) we have that

$$x_1(n) = a^n u(n) \overset{z}{\longleftrightarrow} X_1(z) = \frac{1}{1 - az^{-1}}, \qquad \text{ROC: } |z| > |a|$$

Thus, by using (3.2.14), we obtain

$$na^n u(n) \overset{z}{\longleftrightarrow} X(z) = -z\frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad \text{ROC: } |z| > |a| \tag{3.2.15}$$

If we set $a = 1$ in (3.2.15), we find the z-transform of the unit ramp signal

$$nu(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}, \qquad \text{ROC: } |z| > 1 \tag{3.2.16}$$

EXAMPLE 3.2.8

Determine the signal $x(n)$ whose z-transform is given by

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|$$

Solution. By taking the first derivative of $X(z)$, we obtain

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}$$

Thus

$$-z\frac{dX(z)}{dz} = az^{-1}\left[\frac{1}{1-(-a)z^{-1}}\right], \qquad |z| > |a|$$

The inverse z-transform of the term in brackets is $(-a)^n$. The multiplication by z^{-1} implies a time delay by one sample (time-shifting property), which results in $(-a)^{n-1}u(n-1)$. Finally, from the differentiation property we have

$$nx(n) = a(-a)^{n-1}u(n-1)$$

or

$$x(n) = (-1)^{n+1}\frac{a^n}{n}u(n-1)$$

Convolution of two sequences. If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z) \tag{3.2.17}$$

The ROC of $X(z)$ is, at least, the intersection of that for $X_1(z)$ and $X_2(z)$.

Proof The convolution of $x_1(n)$ and $x_2(n)$ is defined as

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

The z-transform of $x(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\right]z^{-n}$$

Upon interchanging the order of the summations and applying the time-shifting property in (3.2.5), we obtain

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n}\right]$$

$$= X_2(z)\sum_{k=-\infty}^{\infty} x_1(k)z^{-k} = X_2(z)X_1(z)$$

EXAMPLE 3.2.9

Compute the convolution $x(n)$ of the signals

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

Solution. From (3.1.1), we have

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

According to (3.2.17), we carry out the multiplication of $X_1(z)$ and $X_2(z)$. Thus

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Hence

$$x(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$$

The same result can also be obtained by noting that

$$X_1(z) = (1 - z^{-1})^2$$

$$X_2(z) = \frac{1 - z^{-6}}{1 - z^{-1}}$$

Then

$$X(z) = (1 - z^{-1})(1 - z^{-6}) = 1 - z^{-1} - z^{-6} + z^{-7}$$

The reader is encouraged to obtain the same result explicitly by using the convolution summation formula (time-domain approach).

The convolution property is one of the most powerful properties of the z-transform because it converts the convolution of two signals (time domain) to multiplication of their transforms. Computation of the convolution of two signals, using the z-transform, requires the following steps:

1. Compute the z-transforms of the signals to be convolved.

$$X_1(z) = Z\{x_1(n)\}$$

$$\qquad\qquad (\text{time domain} \longrightarrow z\text{-domain})$$

$$X_2(z) = Z\{x_2(n)\}$$

2. Multiply the two z-transforms.

$$X(z) = X_1(z)X_2(z), \qquad (z\text{-domain})$$

3. Find the inverse z-transform of $X(z)$.

$$x(n) = Z^{-1}\{X(z)\}, \qquad (z\text{-domain} \longrightarrow \text{time domain})$$

This procedure is, in many cases, computationally easier than the direct evaluation of the convolution summation.

Correlation of two sequences. If

$$x_1(n) \stackrel{z}{\longleftrightarrow} X_1(z)$$

$$x_2(n) \stackrel{z}{\longleftrightarrow} X_2(z)$$

then

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \stackrel{z}{\longleftrightarrow} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1}) \tag{3.2.18}$$

Proof We recall that

$$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$$

Using the convolution and time-reversal properties, we easily obtain

$$R_{x_1x_2}(z) = Z\{x_1(l)\}Z\{x_2(-l)\} = X_1(z)X_2(z^{-1})$$

The ROC of $R_{x_1x_2}(z)$ is at least the intersection of that for $X_1(z)$ and $X_2(z^{-1})$.

As in the case of convolution, the crosscorrelation of two signals is more easily done via polynomial multiplication according to (3.2.18) and then inverse transforming the result.

EXAMPLE 3.2.10

Determine the autocorrelation sequence of the signal

$$x(n) = a^n u(n), \qquad -1 < a < 1$$

Solution. Since the autocorrelation sequence of a signal is its correlation with itself, (3.2.18) gives

$$R_{xx}(z) = Z\{r_{xx}(l)\} = X(z)X(z^{-1})$$

From (3.2.2) we have

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad \text{ROC: } |z| > |a| \qquad \text{(causal signal)}$$

and by using (3.2.15), we obtain

$$X(z^{-1}) = \frac{1}{1 - az}, \qquad \text{ROC: } |z| < \frac{1}{|a|} \qquad \text{(anticausal signal)}$$

Thus

$$R_{xx}(z) = \frac{1}{1 - az^{-1}} \frac{1}{1 - az} = \frac{1}{1 - a(z + z^{-1}) + a^2}, \qquad \text{ROC: } |a| < |z| < \frac{1}{|a|}$$

Since the ROC of $R_{xx}(z)$ is a ring, $r_{xx}(l)$ is a two-sided signal, even if $x(n)$ is causal.

To obtain $r_{xx}(l)$, we observe that the z-transform of the sequence in Example 3.1.5 with $b = 1/a$ is simply $(1 - a^2)R_{xx}(z)$. Hence it follows that

$$r_{xx}(l) = \frac{1}{1 - a^2} a^{|l|}, \qquad -\infty < l < \infty$$

The reader is encouraged to compare this approach with the time-domain solution of the same problem given in Section 2.6.

Multiplication of two sequences. If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n)x_2(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{2\pi j} \oint_C X_1(v) X_2\left(\frac{z}{v}\right) v^{-1}\, dv \tag{3.2.19}$$

where C is a closed contour that encloses the origin and lies within the region of convergence common to both $X_1(v)$ and $X_2(1/v)$.

Proof The z-transform of $x_3(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x_1(n) x_2(n) z^{-n}$$

Let us substitute the inverse transform

$$x_1(n) = \frac{1}{2\pi j} \oint_C X_1(v) v^{n-1} dv$$

for $x_1(n)$ in the z-transform $X(z)$ and interchange the order of summation and integration. Thus we obtain

$$X(z) = \frac{1}{2\pi j} \oint_C X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2(n) \left(\frac{z}{v}\right)^{-n} \right] v^{-1} dv$$

The sum in the brackets is simply the transform $X_2(z)$ evaluated at z/v. Therefore,

$$X(z) = \frac{1}{2\pi j} \oint_C X_1(v) X_2\left(\frac{z}{v}\right) v^{-1}\, dv$$

which is the desired result.

To obtain the ROC of $X(z)$ we note that if $X_1(v)$ converges for $r_{1l} < |v| < r_{1u}$ and $X_2(z)$ converges for $r_{2l} < |z| < r_{2u}$, then the ROC of $X_2(z/v)$ is

$$r_{2l} < \left|\frac{z}{v}\right| < r_{2u}$$

Hence the ROC for $X(z)$ is at least

$$r_{1l}r_{2l} < |z| < r_{1u}r_{2u} \tag{3.2.20}$$

Although this property will not be used immediately, it will prove useful later, especially in our treatment of filter design based on the window technique, where we multiply the impulse response of an IIR system by a finite-duration "window" which serves to truncate the impulse response of the IIR system.

For complex-valued sequences $x_1(n)$ and $x_2(n)$ we can define the product sequence as $x(n) = x_1(n)x_2^*(n)$. Then the corresponding complex convolution integral becomes

$$x(n) = x_1(n)x_2^*(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{z^*}{v^*}\right)v^{-1}\,dv \tag{3.2.21}$$

The proof of (3.2.21) is left as an exercise for the reader.

Parseval's relation. If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}\,dv \tag{3.2.22}$$

provided that $r_{1l}r_{2l} < 1 < r_{1u}r_{2u}$, where $r_{1l} < |z| < r_{1u}$ and $r_{2l} < |z| < r_{2u}$ are the ROC of $X_1(z)$ and $X_2(z)$. The proof of (3.2.22) follows immediately by evaluating $X(z)$ in (3.2.21) at $z = 1$.

The Initial Value Theorem. If $x(n)$ is *causal* [i.e., $x(n) = 0$ for $n < 0$], then

$$x(0) = \lim_{z\to\infty} X(z) \tag{3.2.23}$$

Proof Since $x(n)$ is causal, (3.1.1) gives

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Obviously, as $z \to \infty$, $z^{-n} \to 0$ since $n > 0$, and (3.2.23) follows.

TABLE 3.2 Properties of the z-Transform

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\dfrac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}\,dv$	At least, $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

All the properties of the z-transform presented in this section are summarized in Table 3.2 for easy reference. They are listed in the same order as they have been introduced in the text. The conjugation properties and Parseval's relation are left as exercises for the reader.

We have now derived most of the z-transforms that are encountered in many practical applications. These z-transform pairs are summarized in Table 3.3 for easy reference. A simple inspection of this table shows that these z-transforms are all *rational functions* (i.e., ratios of polynomials in z^{-1}). As will soon become apparent, rational z-transforms are encountered not only as the z-transforms of various important signals but also in the characterization of discrete-time linear time-invariant systems described by constant-coefficient difference equations.

TABLE 3.3 Some Common z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin\omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n\sin\omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

3.3 Rational z-Transforms

As indicated in Section 3.2, an important family of z-transforms are those for which $X(z)$ is a rational function, that is, a ratio of two polynomials in z^{-1} (or z). In this section we discuss some very important issues regarding the class of rational z-transforms.

3.3.1 Poles and Zeros

The *zeros* of a z-transform $X(z)$ are the values of z for which $X(z) = 0$. The *poles* of a z-transform are the values of z for which $X(z) = \infty$. If $X(z)$ is a rational function, then

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{3.3.1}$$

If $a_0 \neq 0$ and $b_0 \neq 0$, we can avoid the negative powers of z by factoring out the terms $b_0 z^{-M}$ and $a_0 z^{-N}$ as follows:

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 z^{-M}}{a_0 z^{-N}} \frac{z^M + (b_1/b_0)z^{M-1} + \cdots + b_M/b_0}{z^N + (a_1/a_0)z^{N-1} + \cdots + a_N/a_0}$$

Since $B(z)$ and $A(z)$ are polynomials in z, they can be expressed in factored form as

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

$$X(z) = Gz^{N-M} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)} \tag{3.3.2}$$

where $G \equiv b_0/a_0$. Thus $X(z)$ has M finite zeros at $z = z_1, z_2, \ldots, z_M$ (the roots of the numerator polynomial), N finite poles at $z = p_1, p_2, \ldots, p_N$ (the roots of the denominator polynomial), and $|N - M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin $z = 0$. Poles or zeros may also occur at $z = \infty$. A zero exists at $z = \infty$ if $X(\infty) = 0$ and a pole exists at $z = \infty$ if $X(\infty) = \infty$. If we count the poles and zeros at zero and infinity, we find that $X(z)$ has exactly the same number of poles as zeros.

We can represent $X(z)$ graphically by a *pole–zero plot* (or *pattern*) in the complex plane, which shows the location of poles by crosses ($\times$) and the location of zeros by circles ($\circ$). The multiplicity of multiple-order poles or zeros is indicated by a number close to the corresponding cross or circle. Obviously, by definition, the ROC of a z-transform should not contain any poles.

EXAMPLE 3.3.1

Determine the pole–zero plot for the signal

$$x(n) = a^n u(n), \qquad a > 0$$

Solution. From Table 3.3 we find that

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad \text{ROC: } |z| > a$$

Thus $X(z)$ has one zero at $z_1 = 0$ and one pole at $p_1 = a$. The pole–zero plot is shown in Fig. 3.3.1. Note that the pole $p_1 = a$ is not included in the ROC since the z-transform does not converge at a pole.

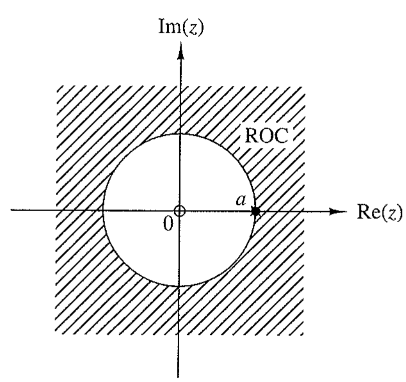


Figure 3.3.1
Pole–zero plot for the causal exponential signal $x(n) = a^n u(n)$.

EXAMPLE 3.3.2

Determine the pole–zero plot for the signal

$$x(n) = \begin{cases} a^n, & 0 \le n \le M-1 \\ 0, & \text{elsewhere} \end{cases}$$

where $a > 0$.

Solution. From the definition (3.1.1) we obtain

$$X(z) = \sum_{n=0}^{M-1} (az^{-1})^n = \frac{1-(az^{-1})^M}{1-az^{-1}} = \frac{z^M - a^M}{z^{M-1}(z-a)}$$

Since $a > 0$, the equation $z^M = a^M$ has M roots at

$$z_k = ae^{j2\pi k/M} \qquad k = 0, 1, \ldots, M-1$$

The zero $z_0 = a$ cancels the pole at $z = a$. Thus

$$X(z) = \frac{(z-z_1)(z-z_2)\cdots(z-z_{M-1})}{z^{M-1}}$$

which has $M-1$ zeros and $M-1$ poles, located as shown in Fig. 3.3.2 for $M = 8$. Note that the ROC is the entire z-plane except $z = 0$ because of the $M-1$ poles located at the origin.

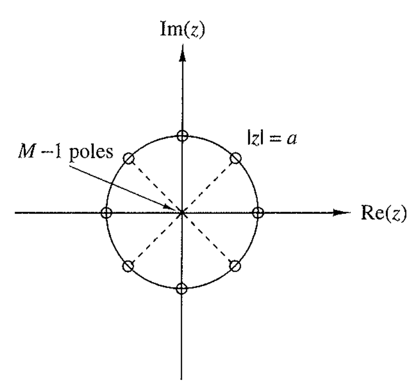


Figure 3.3.2 Pole–zero pattern for the finite-duration signal $x(n) = a^n$, $0 \le n \le M-1 (a > 0)$, for $M = 8$.

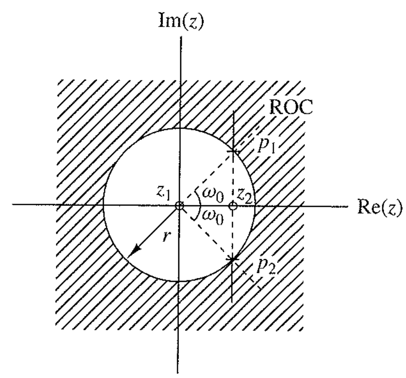


Figure 3.3.3
Pole-zero pattern for Example 3.3.3.

Clearly, if we are given a pole–zero plot, we can determine $X(z)$, by using (3.3.2), to within a scaling factor G. This is illustrated in the following example.

EXAMPLE 3.3.3

Determine the z-transform and the signal that corresponds to the pole–zero plot of Fig. 3.3.3.

Solution. There are two zeros ($M = 2$) at $z_1 = 0$, $z_2 = r\cos\omega_0$ and two poles ($N = 2$) at $p_1 = re^{j\omega_0}$, $p_2 = re^{-j\omega_0}$. By substitution of these relations into (3.3.2), we obtain

$$X(z) = G\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = G\frac{z(z - r\cos\omega 0)}{(z-re^{j\omega_0})(z-re^{-j\omega_0}}), \qquad \text{ROC: } |z| > r$$

After some simple algebraic manipulations, we obtain

$$X(z) = G\frac{1 - rz^{-1}\cos\omega 0}{1 - 2rz^{-1}\cos\omega_0 + r^2z^{-2}}, \qquad \text{ROC: } |z| > r$$

From Table 3.3 we find that

$$x(n) = G(r^n \cos\omega_0 n)u(n)$$

From Example 3.3.3, we see that the product $(z - p_1)(z - p_2)$ results in a polynomial with real coefficients, when p_1 and p_2 are complex conjugates. In general, if a polynomial has real coefficients, its roots are either real or occur in complex-conjugate pairs.

As we have seen, the z-transform $X(z)$ is a complex function of the complex variable $z = \Re(z) + j\Im(z)$. Obviously, $|X(z)|$, the magnitude of $X(z)$, is a real and positive function of z. Since z represents a point in the complex plane, $|X(z)|$ is a two-dimensional function and describes a "surface." This is illustrated in Fig. 3.3.4 for the z-transform

$$X(z) = \frac{z^{-1} - z^{-2}}{1 - 1.2732z^{-1} + 0.81z^{-2}} \tag{3.3.3}$$

which has one zero at $z_1 = 1$ and two poles at $p_1,\ p_2 = 0.9e^{\pm j\pi/4}$. Note the high peaks near the singularities (poles) and the deep valley close to the zero.