Summary Page for Pole-Zero Cancellation for Filtering Data with Finite-Length Sineware

System 1: "IIR" with single pole at $e^{j2\pi\frac{\ell}{N}}$

ℓ, integer

$\ell = 0, 1, \ldots, N-1$

$$y[n] = e^{j2\pi\frac{\ell}{N}}\, y[n-1] + x[n] - x[n-N]$$

has same impulse response as FIR filter

System 2: $y[n] = \sum_{k=0}^{N-1} e^{j2\pi\frac{\ell}{N}k}\, x[n-k]$

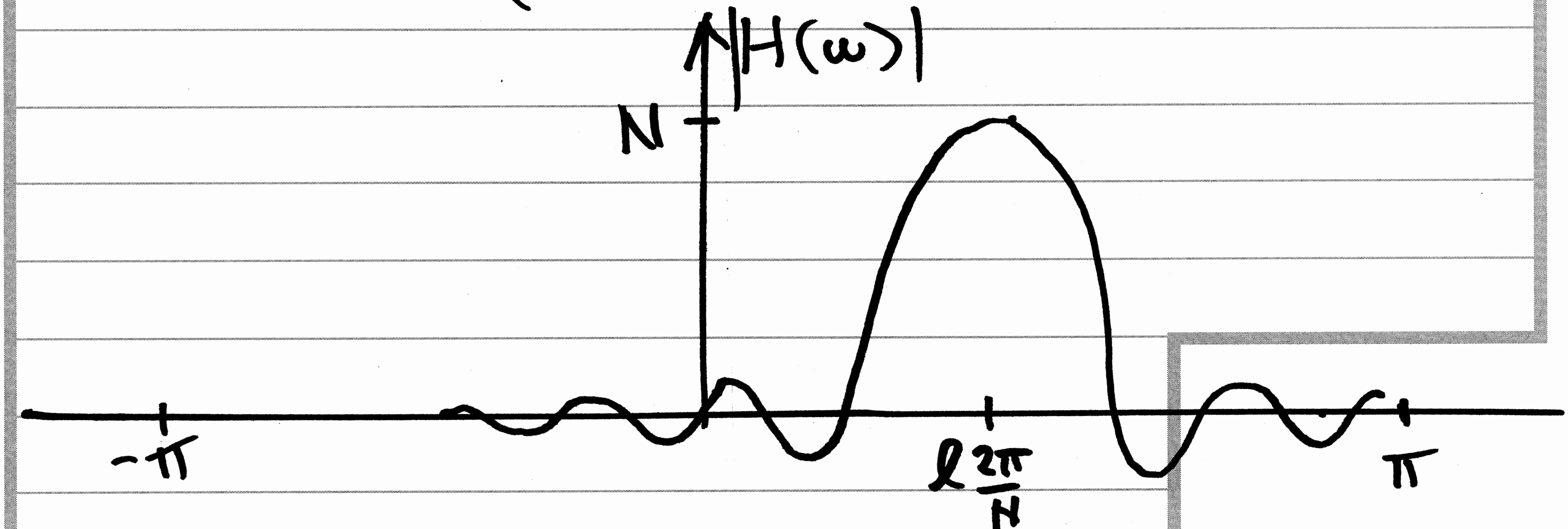
Impulse Response: let $x[n] = \delta[n] \rightarrow y[n] = h[n]$

$$h[n] = e^{0}\delta[n] + e^{j2\pi\frac{\ell}{N}(1)}\delta[n-1] + \ldots + e^{j2\pi\frac{\ell}{N}(N-1)}\delta[n-(N-1)]$$

$$= e^{j2\pi\frac{\ell}{N}n}\left\{u[n] - u[n-N]\right\}$$

Frequency Response = DTFT of h[n]

$$H(\omega) = \frac{\sin\left(\frac{N}{2}\left(\omega - l\frac{2\pi}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - l\frac{2\pi}{N}\right)\right)} e^{-j\frac{(N-1)}{2}\left(\omega - l\frac{2\pi}{N}\right)}$$



$$\left.\begin{array}{ll} H\left(l\frac{2\pi}{N}\right) = N & \\ H\left(l\frac{2\pi}{N} + m\frac{2\pi}{N}\right) = 0 & m \neq l \end{array}\right\} -\pi < \omega < \pi$$

Recall: for infinite-length input sinewaves

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{\begin{array}{c} \text{LTI} \\ h[n] \longleftrightarrow H(\omega) \end{array}} \longrightarrow H(\omega_0)\, e^{j\omega_0 n}$$

for all n

(this result was obtained thru convolution)

Recall:

$$x[n] \longrightarrow \boxed{\begin{array}{c} \text{LTI} \\ h[n] \end{array}} \longrightarrow y[n]$$

$$r_{yx}[\ell] = h[\ell] * r_{xx}[\ell]$$

$$r_{yy}[\ell] = r_{hh}[\ell] * r_{xx}[\ell]$$

What is $r_{hh}[\ell]$ for $h[n] = e^{j 2\pi \frac{\ell'}{N} n} \{u[n] - u[n-N]\}$??

First, what is $r_{xx}[\ell]$ for $x[n] = u[n] - u[n-N]$ (4)

Answer:

$$r_{xx}[\ell] = \{1, 2, \ldots, N-1, \underset{\substack{\uparrow \\ \ell=0}}{N}, N-1, \ldots, 2, 1\}$$

Thus, $r_{hh}[\ell]$ for $h[n] = e^{j 2\pi \frac{\ell'}{N} n} \{u[n] - u[n-N]\}$

$$r_{hh}[\ell] = e^{j 2\pi \frac{\ell'}{N} \ell} \; r_{xx}[\ell] \text{ above}$$

$$\{ e^{-j\frac{2\pi}{N}\ell'(N-1)}, e^{-j\frac{2\pi}{N}\ell'(N-2)} 2, \ldots, \underset{\substack{\uparrow \\ \ell=0}}{N}, e^{j\frac{2\pi}{N}\ell'} (N-1), \ldots, e^{j\frac{2\pi}{N}\ell'(N-2)} 2, e^{j\frac{2\pi}{N}\ell'(N-1)} \}$$

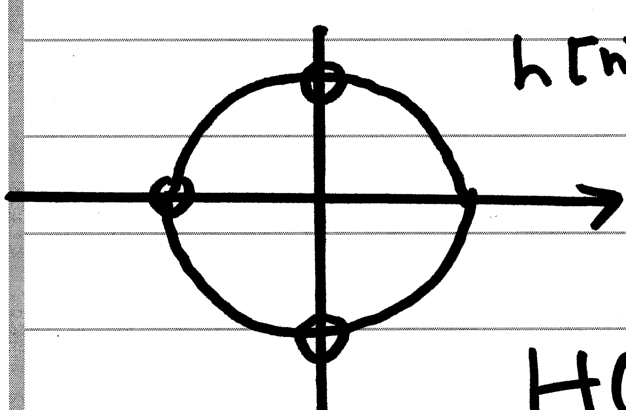
recall: $r_{xx}[-\ell] = r_{xx}^*[\ell]$

Example: $N=4$

$$A: \quad y[n] = y[n-1] + x[n] - x[n-4]$$

$$\left.\begin{array}{l} e^{j\frac{2\pi}{4}(0)} \Rightarrow \ell = 0 \end{array}\right\} \text{pole-zero cancellation at } z=1$$

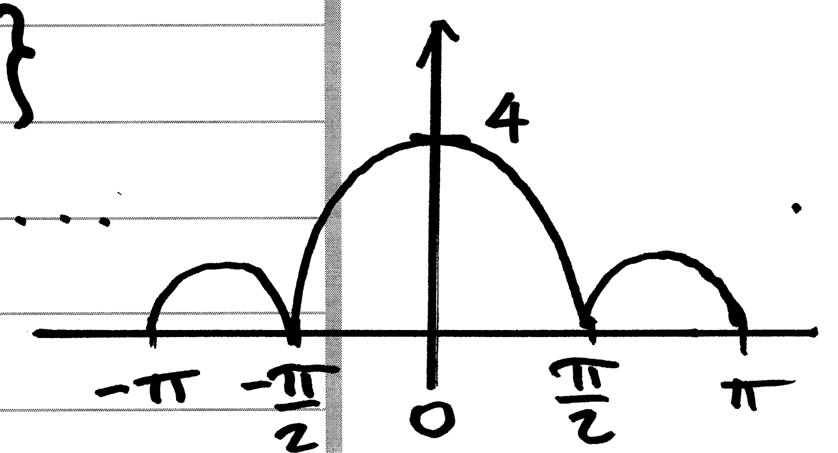
$$h[n] = \{\underset{\uparrow}{1}, 1, 1, 1\}$$



$$H(\omega) = \frac{\sin\left(\frac{4}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{3}{2}\omega}$$

$$H(0) = 4 \qquad H\left(\tfrac{\pi}{2}\right) = H(\pi) = H\left(-\tfrac{\pi}{2}\right) = 0$$

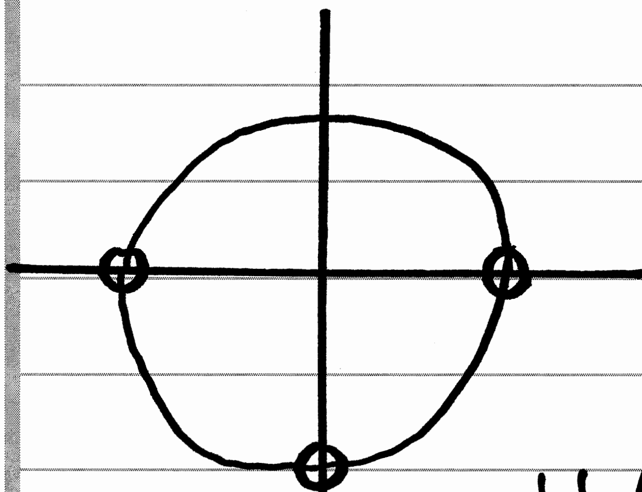
$$r_{hh}[\ell] = \{1, 2, 3, \underset{\underset{\ell=0}{\uparrow}}{4}, 3, 2, 1\}$$



B: $y[n] = j\,y[n-1] + x[n] - x[n-4]$

$\ell = 1$

$$e^{j\frac{2\pi}{4}(1)} = e^{j\frac{\pi}{2}} \Rightarrow \ell = 1 \quad \text{pole-zero cancellation at } z = j$$



$$H(\omega) = \frac{\sin\left(\frac{4}{2}\left(\omega - \frac{\pi}{2}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - \frac{\pi}{2}\right)\right)} e^{-j\frac{3}{2}\left(\omega - \frac{\pi}{2}\right)}$$

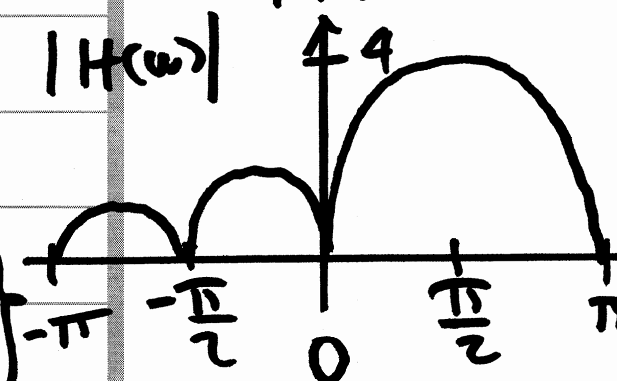
$H\left(\frac{\pi}{2}\right) = 4 \qquad H(0) = H\left(-\frac{\pi}{2}\right) = H(\pi) = 0$

$H(-\pi) = 0$

$$r_{hh}[\ell] = e^{j\frac{\pi}{2}\ell}\{1, 2, 3, 4, 3, 2, 1\}$$

$$* \{j, -1, -j, 1, j, -1, -j\}$$

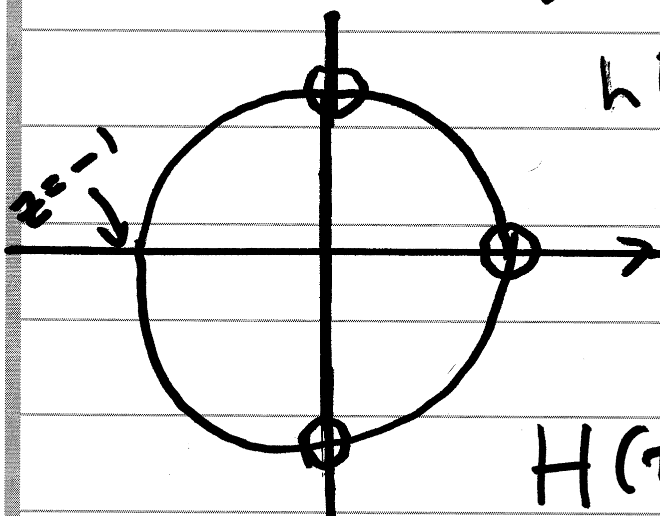
$$= \{j, -2, -3j, \underset{\uparrow \ell = 0}{4}, 3j, -2, -j\}$$



$\underline{C}$: $y[n] = -y[n-1] + x[n] - x[n-4]$

$$e^{j\frac{2\pi}{4}(2)} = e^{j\pi} = -1 \quad \Bigg\} \text{ pole-zero cancellation at } z=-1$$

$\Rightarrow \ell = 2$



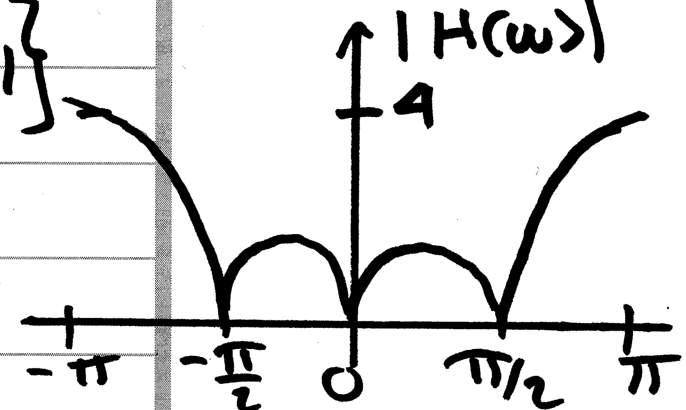
$$h[n] = \{\underset{\uparrow}{1}, -1, 1, -1\}$$

$$H(\omega) = \frac{\sin\left(\frac{4}{2}(\omega-\pi)\right)}{\sin\left(\frac{1}{2}(\omega-\pi)\right)} e^{-j\frac{3}{2}(\omega-\pi)}$$

$H(\pi) = 4 \qquad H(0) = H(\frac{\pi}{2}) = H(-\frac{\pi}{2}) = 0$

$$r_{hh}[\ell] = e^{j\pi\ell} \cdot \{1, 2, 3, 4, 3, 2, 1\}$$

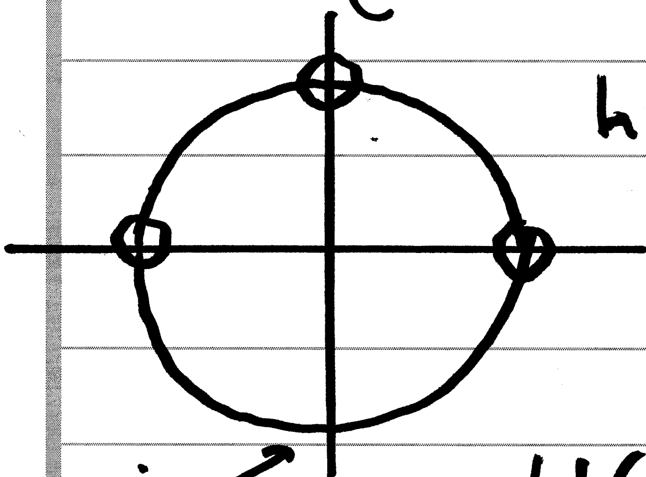
$$= \{-1, 2, -3, \underset{\substack{\uparrow \\ \ell=0}}{4}, -3, 2, -1\}$$



$N=4$

D: $y[n] = -j\,y[n-1] + x[n] - x[n-4]$

$\ell=3 \quad e^{j\frac{2\pi}{4}(3)} = e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}}$ } pole-zero cancellation at $z=-j$



$h[n] = \{\underset{\uparrow}{1}, -j, -1, j\}$

$$H(\omega) = \frac{\sin\left(\frac{4}{2}\left(\omega+\frac{\pi}{2}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega+\frac{\pi}{2}\right)\right)} e^{-j\frac{3}{2}\left(\omega+\frac{\pi}{2}\right)}$$

$H\left(-\frac{\pi}{2}\right) = 4 \qquad H(0) = H\left(\frac{\pi}{2}\right) = H(\pi) = 0$

$$r_{hh}[\ell] = e^{-j\frac{\pi}{2}\ell}\{1, 2, 3, 4, 3, 2, 1\}$$

$$= \{-j, -2, 3j, \underset{\underset{\ell=0}{\uparrow}}{4}, -3j, -2, j\}$$