Summary Page for Pole-Zero Cancellation for Filtering Data with Finite-Length Sineware

System 1: "IIR" with single pole at $e^{j2\pi\frac{\ell}{N}}$

ℓ, integer

$\ell = 0, 1, \ldots, N-1$

$$y[n] = e^{j2\pi\frac{\ell}{N}} y[n-1] + x[n] - x[n-N]$$

has same impulse response as FIR filter

System 2: $y[n] = \sum_{k=0}^{N-1} e^{j2\pi\frac{\ell}{N}k} x[n-k]$

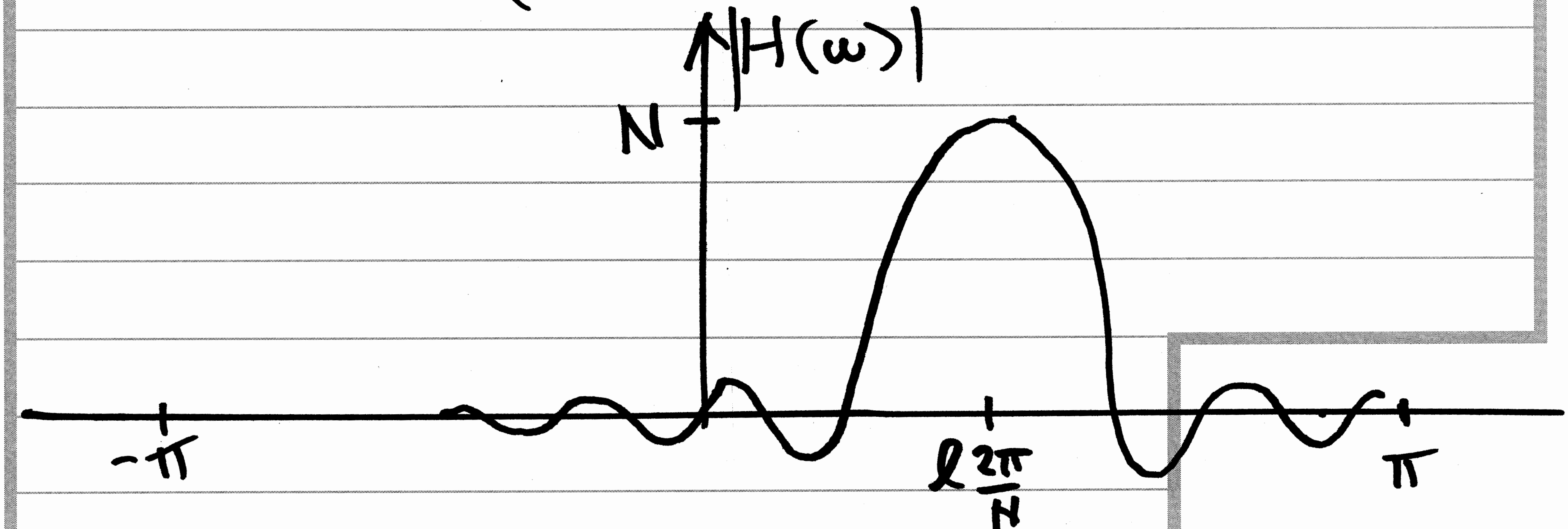
Impulse Response: let $x[n] = \delta[n] \rightarrow y[n] = h[n]$

$$h[n] = e^{0}\delta[n] + e^{j2\pi\frac{\ell}{N}(1)}\delta[n-1] + \ldots + e^{j2\pi\frac{\ell}{N}(N-1)}\delta[n-(N-1)]$$

$$= e^{j2\pi\frac{\ell}{N}n}\{u[n] - u[n-N]\}$$

Frequency Response = DTFT of $h[n]$

$$H(\omega) = \frac{\sin\left(\frac{N}{2}\left(\omega - l\frac{2\pi}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - l\frac{2\pi}{N}\right)\right)} e^{-j\frac{(N-1)}{2}\left(\omega - l\frac{2\pi}{N}\right)}$$



$$\left.\begin{array}{ll} H\left(l\frac{2\pi}{N}\right) = N & \\ H\left(l\frac{2\pi}{N} + m\frac{2\pi}{N}\right) = 0 & m \neq l \end{array}\right\} -\pi < \omega < \pi$$

Recall: for infinite-length input sinewaves

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n] \leftrightarrow H(\omega)\end{matrix}} \longrightarrow H(\omega_0)\, e^{j\omega_0 n}$$

for all n

(this result was obtained thru convolution)

Recall:

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n]\end{matrix}} \longrightarrow y[n]$$

$$r_{yx}[\ell] = h[\ell] * r_{xx}[\ell]$$

$$r_{yy}[\ell] = r_{hh}[\ell] * r_{xx}[\ell]$$

What is $r_{hh}[\ell]$ for $h[n] = e^{j2\pi \frac{\ell'}{N} n} \{u[n] - u[n-N]\}$??

First, what is $r_{xx}[\ell]$ for $x[n] = u[n] - u[n-N]$

Answer:

$$r_{xx}[\ell] = \{1, 2, \ldots, N-1, \underset{\substack{\uparrow \\ \ell=0}}{N}, N-1, \ldots, 2, 1\}$$

Thus, $r_{hh}[\ell]$ for $h[n] = e^{j 2\pi \frac{\ell'}{N} n}\{u[n] - u[n-N]\}$

$r_{hh}[\ell] = e^{j 2\pi \frac{\ell'}{N} \ell}\, r_{xx}[\ell]$ above

$$\{e^{-j\frac{2\pi}{N}\ell'(N-1)}, e^{-j\frac{2\pi}{N}\ell'(N-2)}\,2, \ldots, \underset{\substack{\uparrow \\ \ell=0}}{N}, e^{j\frac{2\pi}{N}\ell'}(N-1), \ldots, e^{j\frac{2\pi}{N}\ell'(N-2)}\,2, e^{j\frac{2\pi}{N}\ell'(N-1)}\}$$

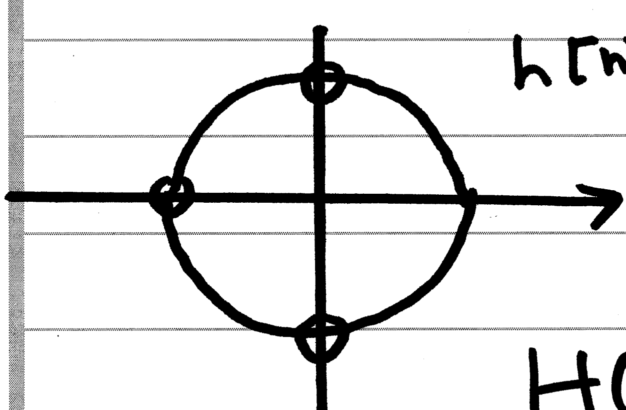
recall: $r_{xx}[-\ell] = r_{xx}^{*}[\ell]$

Example: $N=4$

A: $y[n] = y[n-1] + x[n] - x[n-4]$

$e^{j\frac{2\pi}{4}(0)} \Rightarrow \ell = 0$ } pole-zero cancellation at $z = 1$

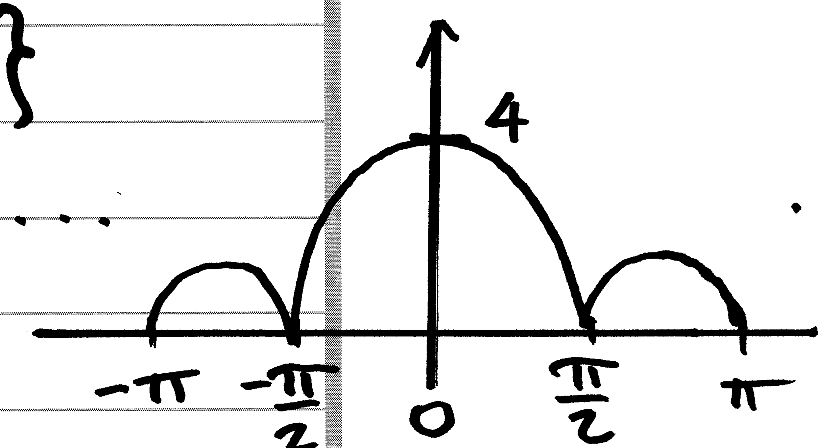
$h[n] = \{\underset{\uparrow}{1}, 1, 1, 1\}$



$$H(\omega) = \frac{\sin\left(\frac{4}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{3}{2}\omega}$$

$H(0) = 4 \qquad H\left(\frac{\pi}{2}\right) = H(\pi) = H\left(-\frac{\pi}{2}\right) = 0$

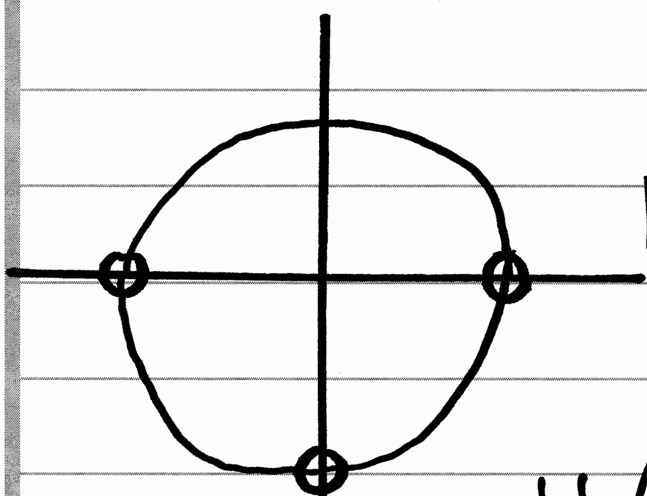
$r_{hh}[\ell] = \{1, 2, 3, \underset{\substack{\uparrow \\ \ell=0}}{4}, 3, 2, 1\}$



B: $y[n] = j\,y[n-1] + x[n] - x[n-4]$

$\ell = 1$

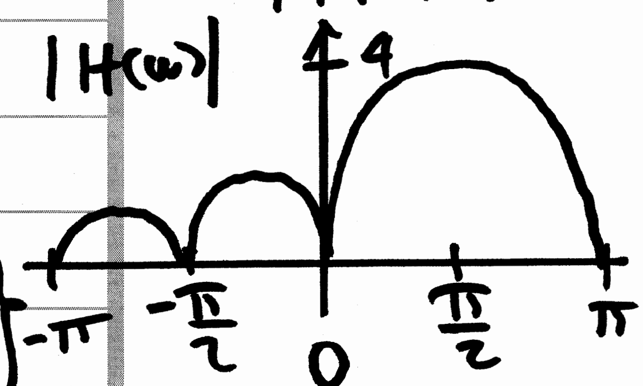
$$e^{j\frac{2\pi}{4}(1)} = e^{j\frac{\pi}{2}} \Rightarrow \ell = 1 \quad \Big\}\ \text{pole-zero cancellation at } z = j$$



$$H(\omega) = \frac{\sin\left(\frac{4}{2}\left(\omega - \frac{\pi}{2}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - \frac{\pi}{2}\right)\right)}\, e^{-j\frac{3}{2}\left(\omega - \frac{\pi}{2}\right)}$$

$$H\left(\frac{\pi}{2}\right) = 4 \qquad H(0) = H\left(-\frac{\pi}{2}\right) = H(\pi) = 0$$

$$H(-\pi) = 0$$



$$r_{hh}[\ell] = e^{j\frac{\pi}{2}\ell}\,\{1, 2, 3, 4, 3, 2, 1\}$$

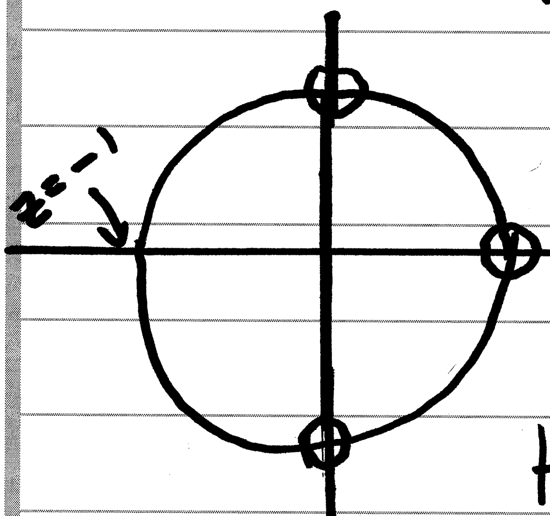
$$* \ \{j, -1, -j, 1, j, -1, -j\}$$

$$= \{j, -2, -3j, \underset{\uparrow\,\ell=0}{4}, 3j, -2, -j\}$$

c: $y[n] = -y[n-1] + x[n] - x[n-4]$

$$\left.e^{j\frac{2\pi}{4}(2)} = e^{j\pi} = -1\right\}$$ pole-zero cancellation at $z = -1$

$\Rightarrow \ell = 2$



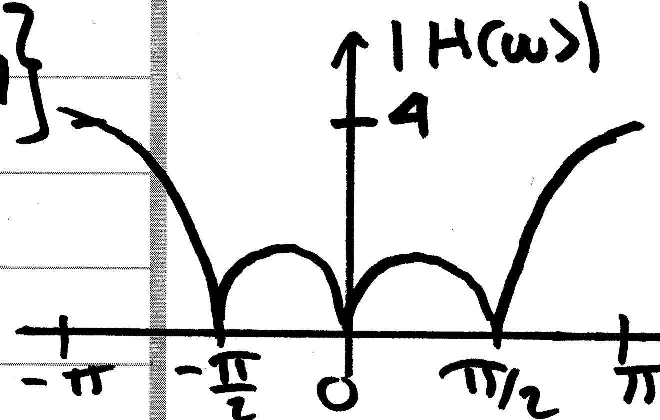
$$h[n] = \{\underset{\uparrow}{1}, -1, 1, -1\}$$

$$H(\omega) = \frac{\sin\left(\frac{4}{2}(\omega-\pi)\right)}{\sin\left(\frac{1}{2}(\omega-\pi)\right)} e^{-j\frac{3}{2}(\omega-\pi)}$$

$H(\pi) = 4 \qquad H(0) = H\left(\frac{\pi}{2}\right) = H\left(-\frac{\pi}{2}\right) = 0$

$$r_{hh}[\ell] = e^{j\pi\ell} \cdot \{1, 2, 3, 4, 3, 2, 1\}$$

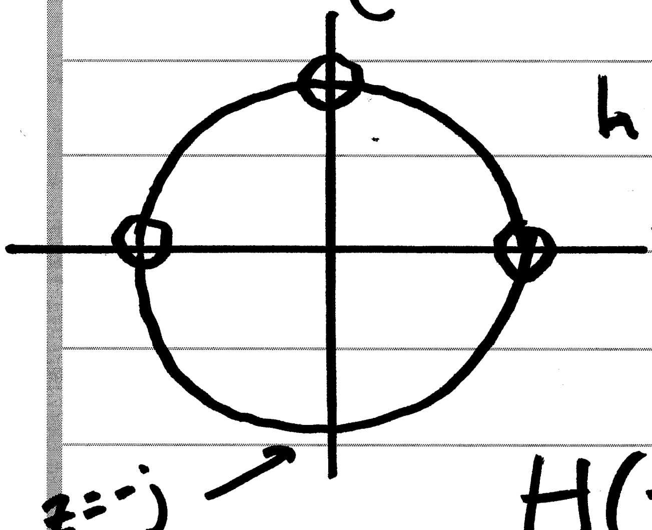
$$= \{-1, 2, -3, \underset{\substack{\uparrow \\ \ell=0}}{4}, -3, 2, -1\}$$



$N=4$

D: $y[n] = -j\, y[n-1] + x[n] - x[n-4]$

$\ell=3 \quad e^{j\frac{2\pi}{4}(3)} = e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}}$ } pole-zero cancellation at $z=-j$



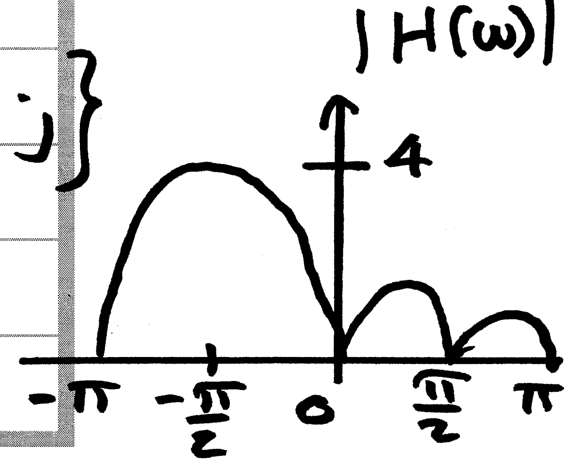
$h[n] = \{\underset{\uparrow}{1}, -j, -1, j\}$

$$H(\omega) = \frac{\sin\left(\frac{4}{2}\left(\omega+\frac{\pi}{2}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega+\frac{\pi}{2}\right)\right)}\, e^{-j\frac{3}{2}\left(\omega+\frac{\pi}{2}\right)}$$

$H\left(-\frac{\pi}{2}\right)=4 \qquad H(0)=H\left(\frac{\pi}{2}\right)=H(\pi)=0$

$r_{hh}[\ell] = e^{-j\frac{\pi}{2}\ell}\{1,2,3,4,3,2,1\}$

$= \{-j,-2,3j,\underset{\substack{\uparrow\\ \ell=0}}{4},-3j,-2,j\}$



$$y[n] = jy[n-1] + x[n] - x[n-4].$$

$$Y(z)(1 - jz^{-1}) = X(z)(1 - z^{-4}).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-4}}{1-jz^{-1}} = \frac{z^4-1}{z^3(z-j)}$$

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$$H(z) = \frac{(z-1)(z-j)(z+1)(z+j)}{z^3(z-j)}$$

$$H(z) = z^{-3}(z-1)(z+1)(z+j) = 1 + jz^{-1} - z^{-2} - jz^{-3}$$

Thus, $h[n] = \{1, j, -1, -j\} = e^{j\frac{\pi}{2}n}(u[n] - u[n-4])$

$y[n] = y[n-1] + x[n] - x[n-4]$. Let $x[n] = \delta[n]$ in which case $y[n] = h[n]$. Thus, the defining eqn for $h[n]$ is:

$h[n] = h[n-1] + \delta[n] - \delta[n-4]$. There are no iniitial conditions and the system is causal so $h[n] = 0$ for $n < 0$.

$n = 0:\ h[0] = h[-1] + \delta[0] - \delta[-4]$ --> $h[0] = 1$. From this point onwards $\delta[n] = 0$

$n = 1:\ h[1] = h[0] + \delta[1] - \delta[-3]$ --> $h[1] = 1$ (since $h[0] = 1$.)

$n = 2:\ h[2] = h[1] + \delta[2] - \delta[-2]$ --> $h[2] = 1$ (since $h[1] = 1$.)

$n = 3:\ h[3] = h[2] + \delta[3] - \delta[-1]$ --> $h[3] = 1$ (since $h[2] = 1$.)

$n = 4:\ h[4] = h[3] + \delta[4] - \delta[0] = 1 - 1 = 0$ --> $h[4] = 0$ (since $h[3] = 1$.). From this point onwards $\delta[n-4] = 0$

$n = 5:\ h[5] = h[4] = 0$ --> $h[5] = 0$ (since $h[4] = 0$.)

Thus, $h[n] = \{1, 1, 1, 1\} = u[n] - u[n-4]$