Sinewave Input to DT LTI System

$$x[n]=e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n]=H(\omega_0)e^{j\omega_0 n}$$

$\forall n$

"turned on" for all time

- This I/O relationship for a complex sinewave input was initially derived via convolution

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]=\sum_{k=-\infty}^{\infty} h[k]\,e^{j\omega_0(n-k)}$$

$$=\sum_{k=-\infty}^{\infty} h[k]\,e^{-j\omega_0 k}\;e^{j\omega_0 n}$$

$$=H(\omega_0)\,e^{j\omega_0 n}$$

where: $h[n] \xleftrightarrow{DTFT} H(\omega)$

- Can also derive same I/O relationship in the frequency domain:

$$x[n] = e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi\,\delta(\omega-\omega_0) = X(\omega)$$

$$-\pi < \omega < \pi$$

- Thus: $Y(\omega) = H(\omega)\,X(\omega)$

$$= H(\omega)\,2\pi\,\delta(\omega-\omega_0)$$

$$= H(\omega_0)\,2\pi\,\delta(\omega-\omega_0)$$

Taking the Inverse DTFT of $Y(\omega)$ yields

$$y[n] = H(\omega_0)\,e^{j\omega_0 n}$$

Sidenotes:

$(j)^n = e^{j\frac{\pi}{2}n}$ $\quad(-1)^n = e^{j\pi n}$ $\quad(-j) = e^{-j\frac{\pi}{2}n}$

$1 = e^{j0n}$