Relationship CTFT-DTFT for Sinewaves

- **Recall:**

$$x_a(t) \xleftrightarrow{\text{CTFT}} X_a(f) \qquad x[n] \xleftrightarrow{\text{DTFT}} X(\omega)$$

where: $x[n] = x_a(nT_s) = x_a\left(\frac{n}{F_s}\right)$

- $F_s = \frac{1}{T_s}$ = sampling rate
- Proved previously: $X(\omega) = X_s\left(F_s \frac{\omega}{2\pi}\right)$
- **where:** $X_s(f) = F_s \sum_{k=-\infty}^{\infty} X_a(f - kF_s)$
- **That is:** $X(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a\left(F_s \frac{\omega}{2\pi} - kF_s\right)$

$$\boxed{\omega = 2\pi \frac{f}{F_s}} \qquad = F_s \sum_{k=-\infty}^{\infty} X_a\left(\frac{F_s}{2\pi}(\omega - k2\pi)\right)$$

$X_s(f)$ period $= F_s \Rightarrow X(\omega)$: period $= 2\pi$

- If $X_a(f) = 0$ for $|f| > W$

and $F_s > 2W$, then

$$X(\omega) = F_s X_a\left(F_s \frac{\omega}{2\pi}\right) \text{ for } -\pi < \omega < \pi$$

- only $k=0$ term in sum contributes in $-\pi < \omega < \pi$
- all other values of k simply serve to make the DTFT be periodic with period 2π which we proved from the "definition" of the DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

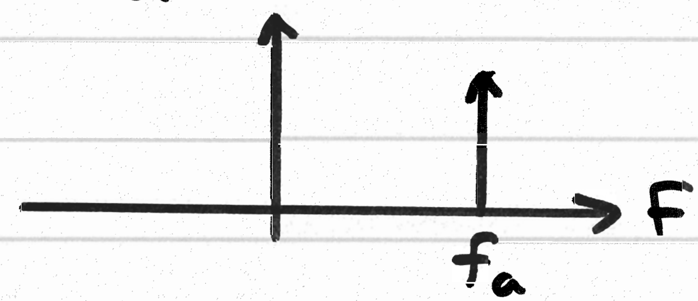
- If $F_s > 2W$, then:

$$\underbrace{-\frac{F_s}{2} < f < \frac{F_s}{2}}_{\text{analog frequency range}} \text{ is mapped to } \underbrace{-\pi < \omega < \pi}_{\text{discrete-time frequencies}}$$

$$\boxed{\text{linear compression} \quad \omega = 2\pi f / F_s}$$

- Consider sampling a sinewave:

$$x_a(t) = e^{j2\pi f_a t} \xleftrightarrow{\text{CTFT}} X_a(f) = \delta(f - f_a)$$



- assume $F_s = \frac{1}{T_s} > 2f_a$

$$x[n] = x_a(nT_s) = x_a(n/F_s) = e^{j2\pi \frac{f_a}{F_s} n} = e^{j\omega_d n}$$

- The relationship between the CT frequency variable and the DT frequency variable pops right out of sampling a sinewave

$$\boxed{\omega_d = 2\pi \frac{f_a}{F_s}} \quad \text{for } -\frac{F_s}{2} < f_a < \frac{F_s}{2}$$

$$x[n] = x_a(n/F_s) = e^{j 2\pi \frac{f_a}{F_s} n} \xleftrightarrow{\text{DTFT}} X(\omega)$$ ④

$$X(\omega) = F_s \, X_a\!\left(F_s \frac{\omega}{2\pi}\right)$$ over $-\pi < \omega < \pi$

assuming $F_s > 2f_a$

$$= F_s \, \delta\!\left(\frac{F_s \omega}{2\pi} - f_a\right)$$

$$= F_s \, \delta\!\left(\frac{F_s}{2\pi}\left(\omega - \frac{2\pi f_a}{F_s}\right)\right)$$

$$= F_s \frac{2\pi}{F_s} \delta(\omega - \omega_d)$$

$$= 2\pi \, \delta(\omega - \omega_d)$$

Special property of Dirac Delta Function:

$$\delta(ax) = \frac{1}{|a|} \delta(x)$$

DTFT pair: $e^{j\omega_d n} \xleftrightarrow{\text{DTFT}} 2\pi\,\delta(\omega - \omega_d)$

for $-\pi < \omega < \pi$

assume $-\pi < \omega_d < \pi$

Specific numbers: $x_a(t) = e^{j 2\pi f_a t}$

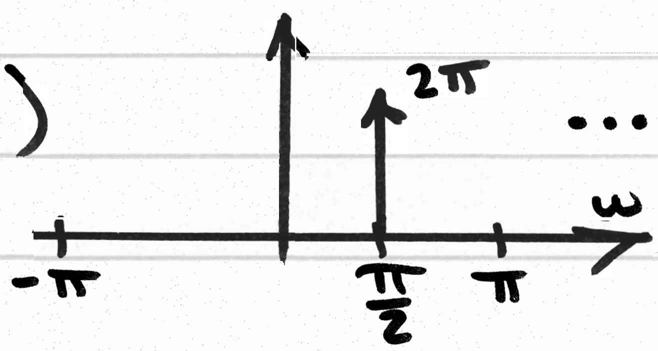
$f_a = 2\ \text{KHz}$

- $F_{s_1} = 8\ \text{KHz} > 2\,(2\ \text{KHz})$

$$x[n] = x_a(n/F_s) = e^{j 2\pi \frac{2K}{8K} n} = e^{j \frac{\pi}{2} n}$$

$$e^{j \frac{\pi}{2} n} \xleftrightarrow{\text{DTFT}} 2\pi\, \delta(\omega - \tfrac{\pi}{2})$$

$$-\pi < \omega < \pi$$

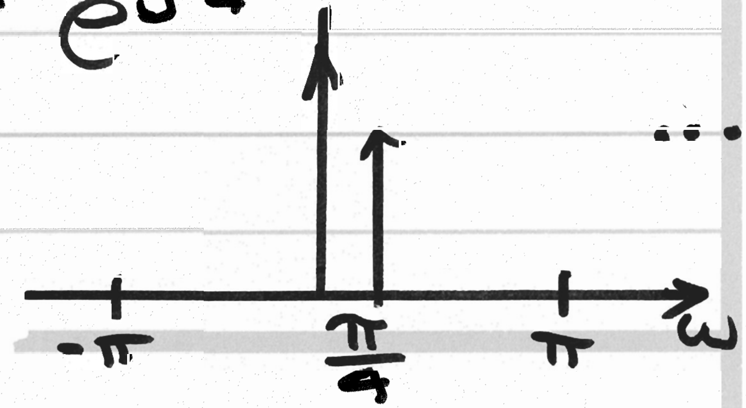


- $F_{s_2} = 16\ \text{KHz}$ (double the sampling rate)

$$x[n] = x_a(n/16K) = e^{j 2\pi \frac{2K}{16K} n} = e^{j \frac{\pi}{4} n}$$

$$e^{j \frac{\pi}{4} n} \xleftrightarrow{\text{DTFT}} 2\pi\, \delta(\omega - \tfrac{\pi}{4})$$

$$-\pi < \omega < \pi$$



double F_s ⇒ compression by factor of 2