**Single Sideband Modulation**

- removing the *negative frequency sideband* in the transmitted spectrum to save bandwidth.

- Consider: \( x(t) \xrightarrow{\mathcal{F}} X(\omega) \) where:

![Diagram](attachment:diagram.png)

- recall: if \( x(t) \) is real-valued: \( X(\omega) = X^*(\omega) \)
  \( \Rightarrow \) \( |X(-\omega)| = |X(\omega)| \) \( \Rightarrow \) magnitude is symmetric about \( \omega = 0 \) \( \Rightarrow \) implies negative frequency content.

- Thus: \( y(t) = 2x(t) \cos(\omega_0 t) \xrightarrow{\mathcal{F}} X(\omega-\omega_0) + X(\omega+\omega_0) \)

![Diagram](attachment:diagram2.png)
To remove the negative frequency sideband, consider the following steps:

- Form $\hat{x}(t)$ by passing $x(t)$ thru an LTI system whose frequency response is:

  $$ h_{HT}(t) = \frac{1}{\pi t} \quad \leftrightarrow \quad H_{HT}(\omega) = \begin{cases} 
  j, & \omega < 0 \\
  -j, & \omega > 0 
\end{cases} $$

  $$ = j u(-\omega) - j u(\omega) $$

- This LTI system is called a Hilbert Transformer.

  $$ x(t) \rightarrow H_{HT}(\omega) \rightarrow \hat{x}(t) $$

- $\hat{x}(t) = x(t) * h_{HT}(t)$

  $$ \hat{x}(\omega) = H_{HT}(\omega) X(\omega) $$

  $$ \hat{x}(\omega) = \begin{cases} 
  j X(\omega), & \omega < 0 \\
  -j X(\omega), & \omega > 0 
\end{cases} $$
Next, form complex-valued signal
\[ \hat{x}(t) = x(t) + j \hat{x}(t) \]

Examine what happens in frequency domain
\[ \hat{X}(\omega) = X(\omega) + j \hat{X}(\omega) \]
\[ = \begin{cases} 
X(\omega) + j (j X(\omega)) , & \text{for } \omega < 0 \\
X(\omega) + j (-j X(\omega)) , & \text{for } \omega > 0 
\end{cases} \]
\[ = \begin{cases} 
(1 - 1) X(\omega) = 0 , & \text{for } \omega < 0 \\
(1 + 1) X(\omega) = 2 X(\omega) , & \text{for } \omega > 0 
\end{cases} \]
\[ \Rightarrow \text{negative frequencies are removed} \]
Thus,
\[ \tilde{x}(t) = x(t) + j \hat{x}(t) \]
\[ \tilde{x}(t) = x(t) + j x(t) * h_{HT}(t) \]
\[ = x(t) * (\delta(t) + j h_{HT}(t)) \]
\[ \tilde{x}(\omega) = X(\omega) + j X(\omega)H_{HT}(\omega) \]
\[ = X(\omega) \left( 1 + j H_{HT}(\omega) \right) \]
\[ \tilde{x}(\omega) \]

- Note, though, that \( \tilde{x}(t) \) is complex-valued and we can only transmit real-valued signals.
\[ n(t) = \tilde{x}(t) e^{j\omega_0 t} \quad \xrightarrow{\mathcal{F}} \quad N(\omega) \]

\[ V(\omega) = \tilde{X}(\omega - \omega_0) \]

- Consider transmitting real part of \( n(t) \)
  \[ y(t) = \text{Re}\{n(t)\} = \frac{1}{2} n(t) + \frac{1}{2} n^*(t) \]

- Note: Property of Fourier Transform
  If \( x(t) \xrightarrow{\mathcal{F}} X(\omega) \), then \( x^*(t) \xrightarrow{\mathcal{F}} X^*(-\omega) \)

Thus,
\[ Y(\omega) = \frac{1}{2} V(\omega) + \frac{1}{2} V^*(-\omega) \]

creates negative frequency content
What is real part of $N(t)$?

\[
y(t) = \text{Re}\{N(t)\} = \text{Re}\{\tilde{x}(t)e^{j\omega_0 t}\}
\]
\[
= \text{Re}\{\left( x(t) + j\hat{x}(t) \right) \left( \cos(\omega_0 t) + j\sin(\omega_0 t) \right) \}
\]= x(t)\cos(\omega_0 t) - \hat{x}(t)\sin(\omega_0 t)
\]

This is the real-valued signal that is transmitted.
Alternative Derivation

\[ y(t) = x(t) \cos(\omega_0 t) - \hat{x}(t) \sin(\omega_0 t) \Rightarrow Y(\omega) = \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0) \]

\[ \hat{x}(t) \sin(\omega_0 t) \Rightarrow \frac{1}{2j} X(\omega - \omega_0) - \frac{1}{2j} X(\omega + \omega_0) \]

Substitute: \[ \hat{x}(\omega) = -j \text{ sgn}(\omega) X(\omega) \]

\[ \hat{x}(t) \sin(\omega_0 t) \Rightarrow -\frac{1}{2} \text{ sgn}(\omega - \omega_0) X(\omega - \omega_0) + \frac{1}{2} \text{ sgn}(\omega + \omega_0) X(\omega + \omega_0) \]

Thus:

\[ Y(\omega) = \frac{1}{2} \left( 1 + \text{ sgn}(\omega - \omega_0) \right) X(\omega - \omega_0) + \frac{1}{2} \left( 1 - \text{ sgn}(\omega + \omega_0) \right) X(\omega + \omega_0) \]

\[ \frac{1}{2} \left( 1 + \text{ sgn}(\omega - \omega_0) \right) = \begin{cases} 1, & \omega > \omega_0 \\ 0, & \omega < \omega_0 \end{cases} \]

\[ \frac{1}{2} \left( 1 - \text{ sgn}(\omega + \omega_0) \right) = \begin{cases} 1, & \omega < -\omega_0 \\ 0, & \omega > \omega_0 \end{cases} \]
Summarizing:

Let:

\[ x(t) \longleftrightarrow \hat{x}(t) \]

real-valued

\[ \hat{x}(t) = x(t) + j x(t) \star h_{HT}(t) \]

where:

\[ h_{HT}(t) = \frac{1}{\pi t} \xrightarrow{\mathcal{F}} -j \text{sgn}(\omega) \]

\[ = j \quad \omega < 0 \]

\[ -j \quad \omega > 0 \]

\[ \hat{x}(t) \xrightarrow{\mathcal{F}} \xi(\omega) \]

complex-valued

\[ \xi(\omega) \xrightarrow{\mathcal{F}} x(t) \]

\[ y(t) = x(t) \cos(\omega_c t) - (x(t) \star h_{HT}(t)) \sin(\omega_c t) \]

real-valued

\[ \omega_c \gg w \]

in contrast to

\[ z(t) = x(t) \cos(\omega t) \]

\[ \xi(\omega) \xrightarrow{\mathcal{F}} z(t) \]

\[ \omega_c \quad \omega_c + w \]

amplitude doubled
Observe: First, recall two trig. identities:
\[ \cos^2(\theta) = \frac{1}{2} + \frac{1}{2} \cos(2\theta) \quad \sin(2\theta) = 2 \sin \theta \cos \theta \]

Again: \( y(t) = x(t) \cos(\omega_c t) - \hat{x}(t) \sin(\omega_c t) \)

\[ \hat{x}(t) = x(t) * \frac{1}{\pi t} \]

Consider: \( \hat{N}(t) = y(t) \cos(\omega_c t) \)

\[ \hat{N}(t) = x(t) \cos^2(\omega_c t) - \hat{x}(t) \cos(\omega_c t) \sin(2\omega_c t) \]

\[ = x(t) (1 + \cos(2\omega_c t)) - \hat{x}(t) \sin(2\omega_c t) \]

\[ = x(t) + \{ x(t) \cos(2\omega_c t) - \hat{x}(t) \sin(2\omega_c t) \} \]

Similar to \( y(t) \) on previous page

But at **double the carrier frequency**

Use LPF to extract \( x(t) \)

(LPF = lowpass filter)
Add’l Note: How to derive the Fourier Transform Pair

\[ h_{HT}(t) = \frac{1}{\pi t} \]

\[ \longleftrightarrow \quad -j \text{sgn}(\omega) \]

\[ \longleftrightarrow \quad j \quad H_{HT}(\omega) \]

Subscript HT =
Hilbert Transformer

\[ H(\omega) = \begin{cases} j, & \omega < 0 \\ -j, & \omega > 0 \end{cases} \]

Derivation: Use FT pair for Unit Step:
\[ u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} + \pi \delta(\omega) \]

Table 4.2

- Plus, two FT properties from Table 4.1:
  - First:
    \[ \text{If: } x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \text{Then: } x_0(t) = \text{odd part} \overset{\mathcal{F}}{\longleftrightarrow} j \text{ Im } \{ X(\omega) \} \]
    \[ = \frac{1}{2} (x(t) - x(-t)) \]
    Thus:
    \[ u(t) - u(-t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{j}{\omega} \left( \frac{1}{2} \right) = -j \frac{\pi}{\omega} \]
  - Second: Duality Prop: \[ X^\ast(\omega) = 2\pi X(-\omega) \]
    Thus:
    \[ -j \frac{\pi}{\omega} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi (-u(\omega) + u(-\omega)) = 2\pi (u(-\omega) - u(\omega)) \]
    Divide by \(-j\pi\) on both sides:
    \[ \frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} \text{j u}(-\omega) - \text{j u}(\omega) = \begin{cases} j, & \omega < 0 \\ -j, & \omega > 0 \end{cases} \]
Figure 8.16 Spectra associated with the frequency-division multiplexing system of Figure 8.15.
Figure 8.19  [Sideband modulated modulating signal with modulation width; (c) spectrum sidebands; (d) spectrum lower sidebands.
In summary, in Sections 8.1 through 8.4 we have seen a number of complex exponential and sinusoidal amplitude modulation. With asynchronous modulation, discussed in Section 8.2.2, a constant must be added to the modulating signal so that it is positive. This results in the presence of the carrier signal as a constant component in the modulated output, requiring more power for transmission, but results in a demodulator that is required in a synchronous system. Alternatively, only one sideband in the modulated output may be retained, which makes it possible to reduce bandwidth and transmitter power, but requires a more sophisticated modulation scheme. Sinusoidal amplitude modulation with both sidebands and the presence of a carrier wave is abbreviated as AM-DSB/SC (amplitude modulation, double sideband/suppressed carrier) when the carrier is suppressed or absent, as AM-DSB/SC (amplitude modulation, double sideband/suppressed carrier). The corresponding single-sideband systems are AM-SSB/SC and AM-SSB/SC.

Sections 8.1 through 8.4 are intended to provide an introduction to the concepts associated with sinusoidal amplitude modulation. There are many other texts and resources available that explore this topic further.
Figure 8.21  System for single-sideband amplitude modulation, using phase-shift network, in which only the lower sidebands are retained.

\[ x(t) \sin(\omega_c t) \xrightarrow{\mathcal{F}} \frac{1}{2j} X(\omega - \omega_c) - \frac{1}{2j} X(\omega + \omega_c) \]

\[ = -j \frac{1}{2} X(\omega - \omega_c) + j \frac{1}{2} X(\omega + \omega_c) \]

spectrum shifted to right by \( \omega_c \), multiplied by \(-j\)

spectrum shifted to left by \( \omega_c \), multiplied by \(+j\)
8.5 AMPLITUDE MODULATION WITH A PULSE-TRAIN CARRIER

8.5.1 Modulation of a Pulse-Train Carrier

In previous sections, we examined amplitude modulation with a sinusoidal carrier. Another important class of amplitude modulation techniques corresponds to the use of a carrier signal that is a pulse train, as illustrated in Figure 8.23; amplitude modulation of this type effectively corresponds to transmitting equally spaced time slices of \(x(t)\). In general, we would not expect that an arbitrary signal could be recovered from such a set of time slices. However, our examination of the concept of sampling in Chapter 7 suggests that this should be possible if \(x(t)\) is band limited and the pulse repetition frequency is high enough.

From Figure 8.23,

\[
y(t) = x(t)c(t);
\]

i.e., the modulated signal \(y(t)\) is the product of \(x(t)\) and the carrier \(c(t)\). With \(Y(j\omega)\), \(X(j\omega)\),

and \(C(j\omega)\) representing the Fourier transforms of each of these signals, it follows from
<table>
<thead>
<tr>
<th>Frequency range</th>
<th>Designation</th>
<th>Typical uses</th>
<th>Propagation method</th>
<th>Channel features</th>
</tr>
</thead>
<tbody>
<tr>
<td>30–300 Hz</td>
<td>ELF</td>
<td>Macrowave, submarine communication</td>
<td>Megametric waves</td>
<td>Penetration of conducting earth and seawater</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(extremely low frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3–3 kHz</td>
<td>VF</td>
<td>Data terminals, telephony</td>
<td>Copper wire</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(voice frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3–30 kHz</td>
<td>VLF</td>
<td>Navigation, telephone, telegraph, frequency and timing standards</td>
<td>Surface ducting (ground wave)</td>
<td>Low attenuation, little fading, extremely stable phase and frequency, large antennas</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(very low frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30–300 kHz</td>
<td>LF</td>
<td>Industrial (power line) communication, aeronautical and maritime long-range navigation, radio beacons</td>
<td>Mostly surface ducting</td>
<td>Slight fading, high atmospheric pulse</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(low frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3–3 MHz</td>
<td>MF</td>
<td>Mobile, AM broadcasting, amateur, public safety</td>
<td>Ducting and ionospheric reflection (sky wave)</td>
<td>Increased fading, but reliable</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(medium frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3–30 MHz</td>
<td>HF</td>
<td>Military communication, aeronautical mobile, international fixed, amateur and citizen's band, industrial</td>
<td>Ionospheric reflecting sky wave, 50–400 km layer altitudes</td>
<td>Intermittent and frequency-selective fading, multipath</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(high frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30–300 MHz</td>
<td>VHF</td>
<td>FM and TV broadcast, land transportation (taxis, buses, railroad)</td>
<td>Sky wave (ionospheric and tropospheric scatter)</td>
<td>Fading, scattering, and multipath</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(very high frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3–3 GHz</td>
<td>UHF</td>
<td>UHF TV, space telemetry, radar, military</td>
<td>Transhorizon tropospheric scatter and line-of-sight relaying</td>
<td>Ionospheric penetration, extraterrestrial noise, high directly</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(ultra high frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3–30 GHz</td>
<td>SHF</td>
<td>Satellite and space communica- tion, common carrier (CC), microwave</td>
<td>Line-of-sight ionosphere penetration</td>
<td>Water vapor and oxygen absorption</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(super high frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30–300 GHz</td>
<td>EHF</td>
<td>Experimental, government, radio astronomy</td>
<td>Line of sight</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(extremely high frequency)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10³–10⁷ GHz</td>
<td>Infrared, visible light, ultraviolet</td>
<td>Optical communications</td>
<td>Line of sight</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 8.18** Allocation of frequencies in the RF spectrum.