Aspects of Solutions for QMF or Two-Channel Perfect Reconstruction Filter Bank

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- Need half-band filter satisfying

$$H_0^2(\omega) - H_0^2(\omega - \pi) = c\, e^{jK\omega}$$

- First, consider an ideal solution:

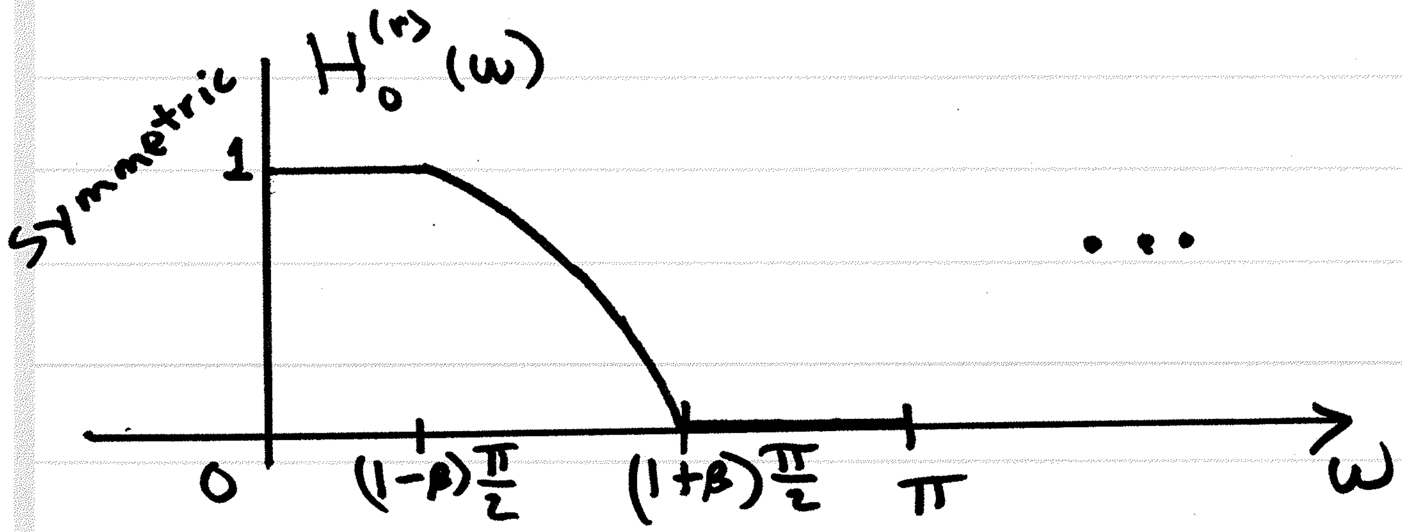
$$H_0^{(r)}(\omega) = \begin{cases} 1, & |\omega| < (1-\beta)\frac{\pi}{2} \\ \cos\left(\frac{1}{2\beta}\left(|\omega| - (1-\beta)\frac{\pi}{2}\right)\right), & (1-\beta)\frac{\pi}{2} < |\omega| < (1+\beta)\frac{\pi}{2} \\ 0, & (1+\beta)\frac{\pi}{2} < |\omega| < \pi \end{cases}$$

where: $0 < \beta < 1$ is roll-off factor

THEN: $H_0(\omega) = H_0^{(r)}(\omega)\, e^{j\frac{\omega}{2}}$

- All together:

$$H_0(\omega) = \begin{cases} e^{j\frac{\omega}{2}}, & |\omega| < (1-\beta)\frac{\pi}{2} \\ e^{j\frac{\omega}{2}} \cos\left(\frac{1}{2\beta}\left(|\omega| - (1-\beta)\frac{\pi}{2}\right)\right) & \text{for } (1-\beta)\frac{\pi}{2} < |\omega| < (1+\beta)\frac{\pi}{2} \\ 0, & (1+\beta)\frac{\pi}{2} < |\omega| < \pi \end{cases}$$



• Show that this satisfies requirement: 3

For $0 < \omega < (1-\beta)\frac{\pi}{2}$:

$$H_0^2(\omega) - \underbrace{H_0^2(\omega-\pi)}_{\text{this is zero for } 0<\omega<(1-\beta)\frac{\pi}{2}}$$

THUS: $H_0^2(\omega) - H_0^2(\omega-\pi) = e^{j\frac{\omega}{2}\cdot 2} = e^{j\omega}$

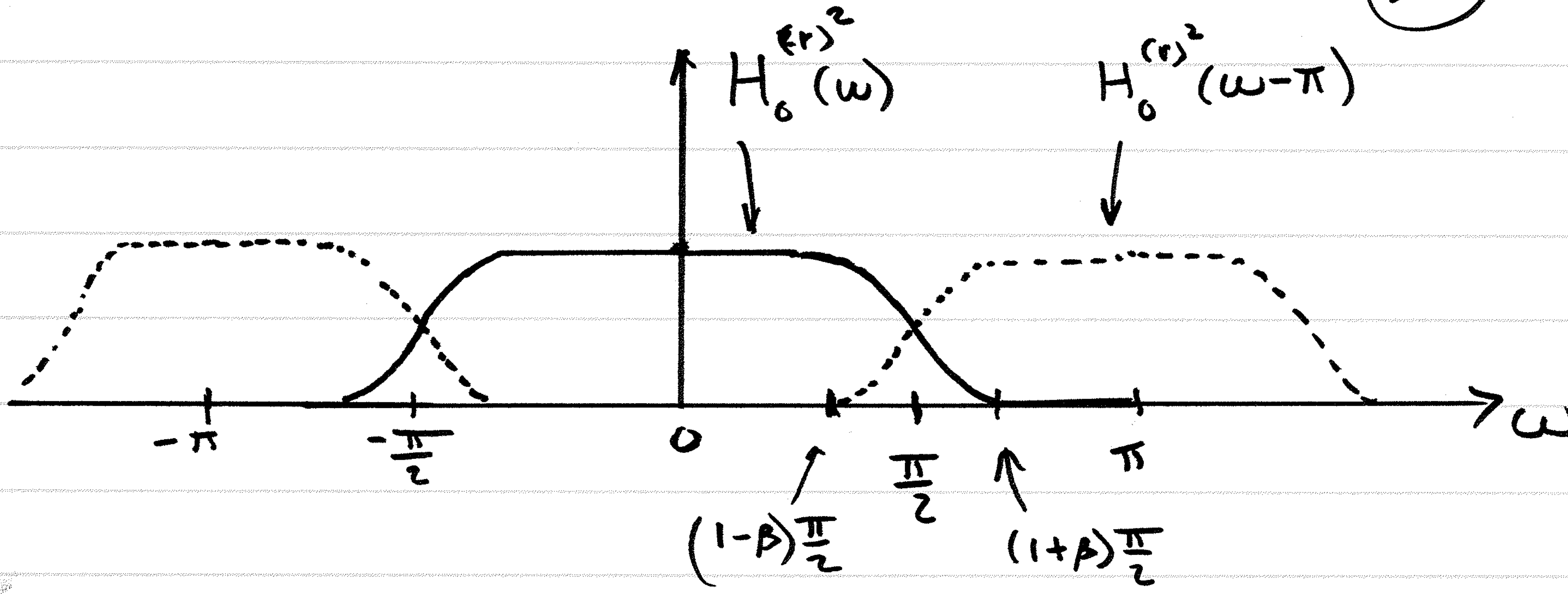
For: $(1-\beta)\frac{\pi}{2} < \omega < (1+\beta)\frac{\pi}{2}$:

$$e^{j\frac{\omega}{2}\cdot 2}\cos^2\left(\frac{1}{2\beta}\left(\omega-(1-\beta)\frac{\pi}{2}\right)\right)$$

$$-e^{j\frac{(\omega-\pi)}{2}\cdot 2}\cos^2\left(\frac{1}{2\beta}|\omega-\pi|-(1-\beta)\frac{\pi}{2}\right)$$

$$= e^{j\omega}\left\{\cos^2\left(\frac{1}{2\beta}\left(\omega-(1-\beta)\frac{\pi}{2}\right)\right)+\cos^2\left(\frac{1}{2\beta}|\omega-\pi|-(1-\beta)\frac{\pi}{2}\right)\right\}$$

(3a)



$$H_0^2(\omega-\pi) = 0 \quad \text{for} \quad 0 < \omega < (1-\beta)\frac{\pi}{2}$$

$$H_0^2(\omega) = 0 \quad \text{for} \quad (1+\beta)\frac{\pi}{2} < \omega < \pi$$

$$\Rightarrow \cos^2\left(\frac{1}{2\beta}\left(\omega-(1-\beta)\frac{\pi}{2}\right)\right)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{1}{\beta}\left(\omega-(1-\beta)\frac{\pi}{2}\right)\right) \quad (4)$$

$$=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2}+\left(\frac{\omega}{\beta}-\frac{\pi}{2\beta}\right)\right)$$

$$\Rightarrow \cos^2\left(\frac{1}{2\beta}\left(|\omega-\pi|-(1-\beta)\frac{\pi}{2}\right)\right)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{1}{\beta}\left(|\omega-\pi|-(1-\beta)\frac{\pi}{2}\right)\right)$$

$$=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2}-\left(\frac{\omega}{\beta}+\frac{\pi}{2\beta}-\frac{\pi}{\beta}\right)\right)$$

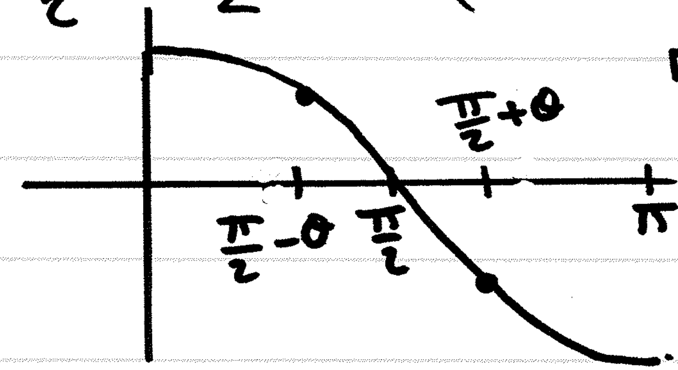
$$=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2}-\left(\frac{\omega}{\beta}-\frac{\pi}{2\beta}\right)\right)$$

where we have used the fact that:

$$|\omega-\pi|=\pi-\omega \quad \text{for} \quad (1-\beta)\frac{\pi}{2}<\omega<(1+\beta)\frac{\pi}{2}$$

· at this point, we have:

$$\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2}+\theta\right)+\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2}-\theta\right)$$



two terms cancel

Thus, we get the sum as 1!

✓ checks

For: $(1+\beta)\frac{\pi}{2} < \omega < \pi$:

$$\underbrace{H_0^2(\omega)}_{\text{this term } =0 \text{ over this range}} - H_0^2(\omega-\pi)$$

$$- e^{j\frac{(\omega-\pi)}{2}\cdot 2}$$

$$- e^{j\omega}e^{-j\pi} = e^{j\omega}$$

✓ checks

- So this works! What is the impulse response for this filter? ⑥
- It was obtained by sampling a CT pulse shape with a Square-Root Raised Cosine Spectrum

$$h_0[n] = p_{SRRC}(t)\Big|_{t = \frac{T_s}{4} + n\frac{T_s}{2}}$$

where: $p_{SRRC}(t) =$

$$= \frac{2\beta}{\pi\ T_s} \frac{\cos\left[(1+\beta)\pi\frac{t}{T_s}\right] + \dfrac{\sin\left((1-\beta)\pi\frac{t}{T_s}\right)}{4\beta\, t/T_s}}{\left[1 - \left(4\beta\frac{t}{T_s}\right)^2\right]}$$

OR: $h_0[n] = p_{SRRC}\left(t + \frac{T_s}{4}\right)\Big|_{t = n\frac{T_s}{2}}$

- sampling every $\frac{T_s}{2}$ secs, then $\frac{T_s}{4}$ corresponds to a shift of a half-sample $\Rightarrow$ this is what gives rise to the linear phase term $e^{j\frac{\omega}{2}}$

$$P_{SRRC}(f) = \alpha \begin{cases} 1, & 0 \le |f| \le \frac{1-\beta}{2T_s} \\ \cos\left[\frac{\pi T_s}{2\beta}\left(|f| - \frac{1-\beta}{2T_s}\right)\right], & \text{for } \frac{1-\beta}{2T_s} \le |f| \le \frac{1+\beta}{2T_s} \\ 0, & |f| > \frac{1+\beta}{2T_s} \end{cases}$$

Since $T_s > 0$:

$$P_{SRRC}(f) = \alpha \cos\left[\frac{\pi}{2\beta}\left(|fT_s| - \frac{1-\beta}{2}\right)\right]$$

$$= \alpha \cos\left[\frac{1}{2\beta}\left(|\pi f T_s| - (1-\beta)\frac{\pi}{2}\right)\right]$$