# Impulse response for simple first-order Difference Equations



$$y[n] = a\,y[n-1] + x[n]$$

- Determine impulse response. (Zero initial conditions)
- When $x[n] = \delta[n]$, $y[n] = h[n]$
- $h[n] = 0$ for $n < 0 \Rightarrow$ causal
- $h[n] = a\,h[n-1] + \delta[n]$ $\Big\}$ iterate through recursively
- $h[0] = a\,\cancelto{0}{h[-1]} + 1 = 1 = a^0$
- $h[1] = a\,h[0] = a(1) = a$
- $h[2] = a\,h[1] = a \cdot a = a^2$

$\vdots$

$h[n] = a\,h[n-1] \Rightarrow a^n = a\,a^{n-1}$

$$\left.\begin{matrix} h[0] = a^0 \\ h[1] = a \\ h[2] = a^2 \\ \vdots \\ h[n] = a\,a^{n-1} \end{matrix}\right\} h[n] = a^n u[n]$$

- For $a = 1 \Rightarrow y[n] = y[n-1] + x[n]$ ②
- impulse response is unit step

$$h[n] = (1)^n u[n] = u[n]$$

- This is the same impulse response as for the system:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- Two different realizations of the same system
- <u>And recall (side-note)</u> that the inverse system has the difference equation $y[n] = x[n] - x[n-1]$
- since the impulse response is $h[n] = \delta[n] - \delta[n-1]$

and:

$$u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n]$$

- **Second Example**

$$y[n] = a\,y[n-1] + x[n] - a^D x[n-D]$$

- Iterate recursively to find impulse response
- $h[n]=0$ for $n<0$ $\Rightarrow$ causal

$$h[n] = a\,h[n-1] + \delta[n] - a^D \delta[n-D]$$

$$h[0] = a\,h[-1] + 1 + 0 = 1$$

$$h[1] = a\,h[0] + 0 + 0 = a$$

$$\vdots$$

$$h[D-1] = a^{D-1} + 0 + 0 = a^{D-1}$$

$$h[D] = a\,h[D-1] - a^D(1)$$

$$= a\,a^{D-1} - a^D = a^D - a^D = 0$$

$$h[D+1] = a\,h[D] + 0 + 0 = 0$$

$$\Rightarrow h[n] = a^n \{u[n] - u[n-D]\}$$

} finite-length geometric sequence

Consider: $D = N$ and $a = e^{j 2\pi \frac{\ell}{N}}$ $\ell \in$ integer

Thus: $a^D = a^N = e^{j 2\pi \frac{\ell}{N} N} = e^{j\pi 2\ell} = (1)^{\ell} = 1$

($\ell$ = integer)

Thus: impulse response of the system

$$y[n] = e^{j 2\pi \frac{\ell}{N}} y[n-1] + x[n] - x[n-N]$$

is:

$$h[n] = e^{j 2\pi \frac{\ell}{N} n} \{u[n] - u[n-N]\}$$

Consider: $N = 4$:

$\ell = 0$: $y[n] = y[n-1] + x[n] - x[n-4] \Rightarrow h_0[n] = u[n] - u[n-4]$

$= \{\underset{\uparrow}{1}, 1, 1, 1\}$

Consider $\ell = 1$: $y[n] = e^{j\frac{2\pi}{4}} y[n-] + x[n] - x[n-4]$

$$= j\, y[n-1] + x[n] - x[n-4]$$

$$h_1[n] = e^{j\frac{2\pi}{4}n} \{u[n] - u[n-4]\} = \{\underset{\uparrow}{1}, j, -1, -j\}$$

Consider $\ell = 2$: $y[n] = e^{j\frac{2\pi}{4}(2)} y[n-1] + x[n] - x[n-4]$

$$= -y[n-1] + x[n] - x[n-4]$$

$$h_2[n] = e^{j\frac{2\pi(2)}{4}n} \{u[n] - u[n-4]\}$$

$$= (-1)^n \{u[n] - u[n-4]\}$$

$$= \{\underset{\uparrow}{1}, -1, 1, -1\}$$

Consider $\ell = 3$: $y[n] = e^{j\frac{2\pi}{4}(3)} y[n-1] + x[n] - x[n-4]$

$$y[n] = -j\, y[n-1] + x[n] - x[n-4]$$

$$h_2[n] = e^{j\frac{2\pi}{4}(3)n} \{u[n] - u[n-4]\}$$

$$= (-j)^n \{u[n] - u[n-4]\}$$

$$= \{\underset{\uparrow}{1}, -j, -1, +j\}$$

Since: $e^{j\frac{3\pi}{2}} = -j$