• Old Exam 1 Problems on Pole-Zero Cancellation ①

• The old problems center on the interesting result that the following two difference equations have the same Transfer Function $H(z)$ due to a pole-zero cancellation

System 1: $y[n] = e^{j2\pi\frac{k}{N}} y[n-1] + x[n] - x[n-N]$

System 2: $y[n] = \sum_{k'=0}^{N-1} e^{j2\pi\frac{k}{N}k'} x[n-k']$

k, integer ; $N > k$ integer

Note: System 1 requires 1 complex multiply per output point, while System 2 requires $N-1$ multiplies per output point

$H_1(z)$ for System 1:

$$H_1(z) = \frac{1 - z^{-N}}{1 - e^{j2\pi\frac{k}{N}} z^{-1}}$$

$H_2(z)$ for System 2:

$$H_2(z) = \sum_{k'=0}^{N-1} e^{j2\pi\frac{k}{N}k'} z^{-k'} = \sum_{k=0}^{N-1} \left(e^{j\frac{2\pi k}{N}} z^{-1}\right)^{k'}$$

$$= \frac{1 - e^{j2\pi\frac{k}{N}N} z^{-N}}{1 - e^{j2\pi\frac{k}{N}} z^{-1}} = \frac{1 - z^{-N}}{1 - e^{j2\pi\frac{k}{N}} z^{-1}}$$

since: $e^{j2\pi\frac{k}{N}N} = \left(e^{j2\pi}\right)^{k} = (1)^{k} = 1$

- Examine $H_1(z)$ more closely:

$$H_1(z) = \frac{z^{-N}\left(z^{N}-1\right)}{z^{-1}\left(z-e^{j2\pi\frac{k}{N}}\right)}$$

- Zeroes are roots of $z^N=1$, which are equi-spaced around the unit circle:

$$z_\ell = e^{j2\pi\frac{\ell}{N}}, \quad \ell = 0, 1, \ldots, N-1$$

- Thus, the pole at $p_k = e^{j2\pi\frac{k}{N}}$ cancels the zero at $z_k = e^{j2\pi\frac{k}{N}}$

Frequency Response:

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{1 - e^{-j\omega N}}{1 - e^{j\frac{2\pi k}{N}} e^{-j\omega}}$$

$$= \frac{e^{-j\frac{N}{2}\omega}}{e^{-j\frac{1}{2}(\omega - \frac{2\pi k}{N})}} \frac{\frac{1}{2j}}{\frac{1}{2j}} \frac{e^{+j\frac{N}{2}\omega} - e^{-j\frac{N}{2}\omega}}{e^{j\frac{1}{2}(\omega - \frac{2\pi k}{N})} - e^{-j\frac{1}{2}(\omega - \frac{2\pi k}{N})}}$$

$$= \frac{e^{-j\frac{N}{2}\omega}}{e^{-j\frac{1}{2}(\omega - 2\pi\frac{k}{N})}} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}(\omega - 2\pi\frac{k}{N})\right)}$$

Note: $e^{-j\frac{N}{2}(\omega - \frac{2\pi k}{N})} = e^{-j\frac{N}{2}\omega} e^{jk\pi} = e^{-j\frac{N}{2}\omega}(-1)^k$

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Similarly: $\sin\left(\frac{N}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)$

$$= \sin\left(\frac{N}{2}\omega - k\pi\right) = (-1)^k \sin\left(\frac{N}{2}\omega\right)$$

• **Since** $(-1)^k (-1)^k = (-1)^{2k} = \left((-1)^2\right)^k = 1$

$$H(\omega) = \frac{e^{-j\frac{N}{2}\left(\omega - 2\pi\frac{k}{N}\right)}}{e^{-j\frac{1}{2}\left(\omega - 2\pi\frac{k}{N}\right)}} \; \frac{\sin\left(\frac{N}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)}$$

$$= e^{-j\frac{(N-1)}{2}\left(\omega - k\frac{2\pi}{N}\right)} \; \frac{\sin\left(\frac{N}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)}$$

• Let's verify this answer by taking the DTFT of the impulse response using a basic DTFT pair and DTFT property

Recall: $$y[n] = \sum_{k'=0}^{N-1} e^{j2\pi \frac{k}{N} k'} x[n-k']$$ (6)

Let $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$$h[n] = \sum_{k'=0}^{N-1} e^{j2\pi \frac{k}{N} k'} \delta[n-k']$$

The coefficients of an FIR difference equation are the values of the impulse response!

Thus: $h[n] = e^{j2\pi \frac{k}{N} n} \{u[n] - u[n-N]\}$

- $H(\omega) \Rightarrow$ use modulation property and DTFT of rectangular window

  $w_R[n] = u[n] - u[n-N]$

Table 4.6 on pg. 291 has DTFT of rectangular window centered at origin $n=0$. Requires N odd.

- Assume N odd, generalize result later
- Since $w_R[n]$ starts at $n=0$, need to shift 2^{nd} entry in Table 4.6 to right by $n_0 = \frac{N-1}{2}$
- Time-Shift Property applied to DTFT pair:

$$u\left[n+\frac{N-1}{2}\right] - u\left[n-\frac{N-1}{2}\right] \xleftrightarrow{\text{DTFT}} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

$\Rightarrow 2^{nd}$ entry in Table 4.6 with $L = \frac{N-1}{2} \Rightarrow N = 2L+1$

$\Rightarrow$ N is "length" of window

- $w_R[n] = u[n] - u[n-N]$ is $\left\{u\left[n+\frac{N-1}{2}\right] - u\left[n-\frac{N-1}{2}\right]\right\}$ shifted to the right by $n_0 = \frac{N-1}{2}$, THUS

$$w_R[n] \xleftrightarrow{\text{DTFT}} W_R(\omega) = e^{-j\frac{(N-1)}{2}\omega} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

- Now apply modulation property
- $h[n] = e^{j 2\pi \frac{k}{N} n} \; w_R[n] \xleftrightarrow{DTFT}$

$$H(\omega) = e^{-j\frac{(N-1)}{2}\left(\omega - 2\pi\frac{k}{N}\right)} \frac{\sin\left(\frac{N}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)}$$

CHECKS! ✓

<u>Note:</u> This result holds for N odd as well

- This a "crude" bandpass filter centered at $\omega_k = 2\pi\frac{k}{N} = k\frac{2\pi}{N}$
- Recall: $H(z) = \frac{z^{-N}}{z^{-1}} \frac{(z^N - 1)}{z - e^{j 2\pi \frac{k}{N}}}$

and discussion on pg. 3

• zeroes are equi-spaced around unit circle at $z_\ell = e^{j\ell\frac{2\pi}{N}}$ for $\ell = 0, 1, \ldots, N-1$, $\ell \neq k$

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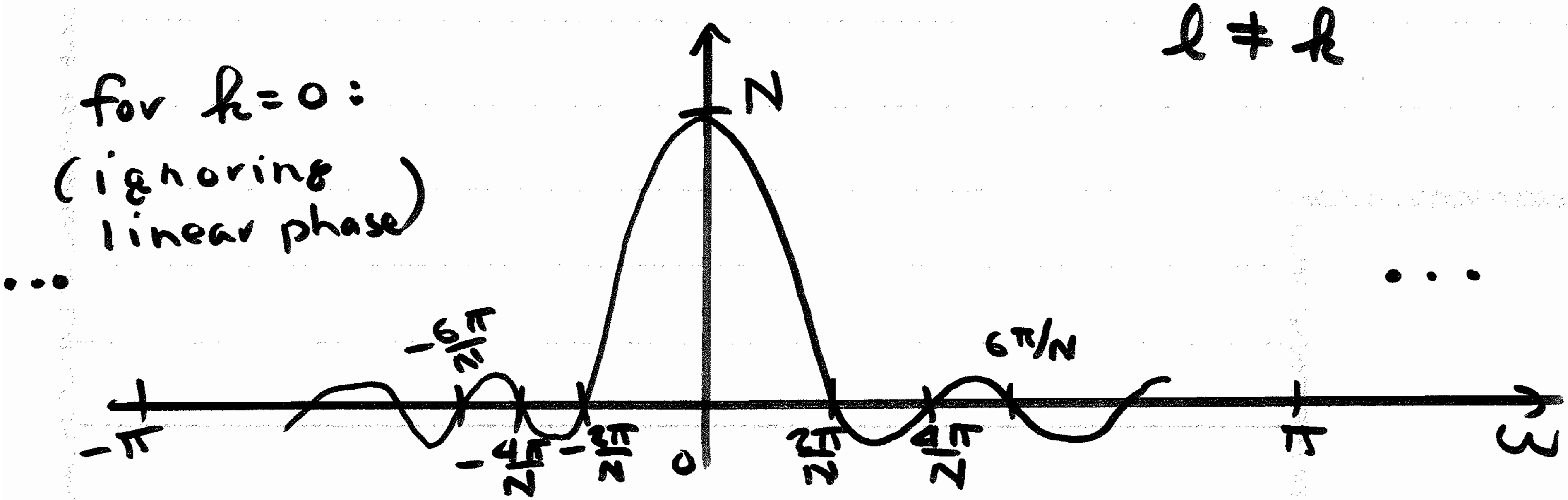
• Sure enough, one can verify that

$H(\omega) = 0$ for $\omega_\ell = \ell\frac{2\pi}{N}$ for $\ell = 0, 1, \ldots, N-1$, $\ell \neq k$

for $k = 0$:
(ignoring linear phase)



plot of $H(\omega)e^{+j\frac{(N-1)}{2}\omega}$

(see Sect. 5.4.5 on Comb Filters)

Time-Domain Method of Determining Impulse Response of Difference Eqn that has a Pole-Zero Cancellation

(3)

$$y[n] = a\,y[n-1] + x[n] - a^D x[n-D]$$

- Iterate recursively to find impulse response
- $h[n]=0$ for $n<0 \Rightarrow$ causal

$$h[n] = a\,h[n-1] + \delta[n] - a^D \delta[n-D]$$

$$h[0] = a\,h[-1] + 1 + 0 = 1$$

$$h[1] = a\,h[0] + 0 + 0 = a$$

$$\vdots$$

$$h[D-1] = a^{D-1} + 0 + 0 = a^{D-1}$$

$$h[D] = a\,h[D-1] - a^D(1)$$

$$= a\,a^{D-1} - a^D = a^D - a^D = 0$$

$$h[D+1] = a\,h[D] + 0 + 0 = 0$$

$$\Rightarrow h[n] = a^n \{u[n] - u[n-D]\}$$ } finite-length geometric sequence