

Parametric Spectral Estimation

- ARMA Spectral Estimation ①
- spectral density is modelled as

$$S_{\text{ARMA}}(\omega) = \frac{\sigma^2 \left| \sum_{k=0}^M b_k e^{-jk\omega} \right|^2}{\left| \sum_{k=0}^N a_k e^{-jk\omega} \right|^2}$$

rational
spectrum

$$a_0 = 1$$

- Notation from statistics literature on ARMA processes $M=Q$, $N=P$
- Indirect spectral estimation: compute $b_k, k=0, \dots, Q$ and $a_k, k=1, \dots, P$ from the estimated autocorrelation values

- only need (theoretically) autocorrelations for lag values $m = 0, 1, \dots, P+q$ to determine $b_k, k = 0, 1, \dots, q$ and $a_k, k = 1, \dots, p$

(note: $r_{xx}[-m] = r_{xx}^*[m]$)

- ARMA spectral estimation works well BUT the ARMA model parameters are nonlinearly related to the autocorrelation values
 - issue of computational complexity

• AR spectral estimation

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$$S_{AR}(\omega) = \frac{\sigma^2}{\left| \sum_{k=0}^p a_k e^{-jk\omega} \right|^2} \quad a_0 = 1$$

• AR model parameters are computed from the estimated/measured autocorrelation values via a linear system of equations of the form

$$\begin{array}{ccc} \underline{R}_{xx} & \underline{a} & = -\underline{r} \\ p \times p & p \times 1 & p \times 1 \end{array}$$

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$$

R_{xx} is a Toeplitz-Hermitian matrix which is formed from

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auto correlations for lag values $0, 1, \dots, p$

$$\begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] & r_{xx}^*[2] & \dots & r_{xx}^*[p-1] \\ r_{xx}[1] & r_{xx}[0] & r_{xx}^*[1] & \dots & r_{xx}^*[p-2] \\ r_{xx}[2] & r_{xx}[1] & r_{xx}[0] & \dots & r_{xx}^*[p-3] \\ \vdots & & & \ddots & \\ r_{xx}[p-1] & r_{xx}[p-2] & & & r_{xx}[0] \end{bmatrix}$$

- Toeplitz: constant along any diagonal
- Hermitian: conjugate-symmetric about main diagonal

• Due to Toeplitz structure of \underline{R}_{xx} (5)

$\underline{R}_{xx} \underline{a} = -\underline{r}$ can be solved with low

complexity \Rightarrow in fact, there is no need

to even form the matrix \underline{R}_{xx}

\Rightarrow Levinson-Durbin algorithm

• Thus, AR spectral estimation is used more in practice than ARMA spectral estimation, even though ARMA generally performs better than AR

• can increase the value of p to make AR perform almost as well as ARMA

So, the steps are:

1. Estimate $r_{xx}[m]$ for lag values $m=0, 1, \dots, P$

$$\hat{r}_{xx}[m] = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x^*[n] x[n+m]$$

or N ✓

$m=0, 1, \dots, P$

2. Solve $\underline{R}_{xx} \underline{a} = -\underline{r}$ for the AR model parameters, via the Levinson-Durbin algorithm $a_k, k=1, \dots, P$

3. Plug the AR model parameters into the AR spectral estimate:

$$S_{xx}(\omega) = \frac{\sigma^2}{\left| 1 + \sum_{k=1}^P a_k e^{-jk\omega} \right|^2}$$

(talk about σ^2 later)

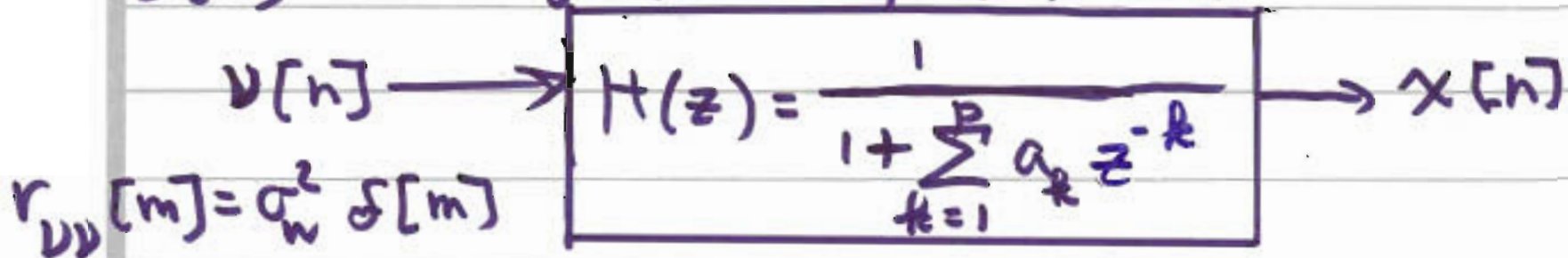
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Relationship between AR model parameters

$a_k, k=1, \dots, P$ and autocorrelation values

$r_{xx}[m], m=0, 1, \dots, P$

Observation: consider passing white noise $v[n]$ through an all-pole filter

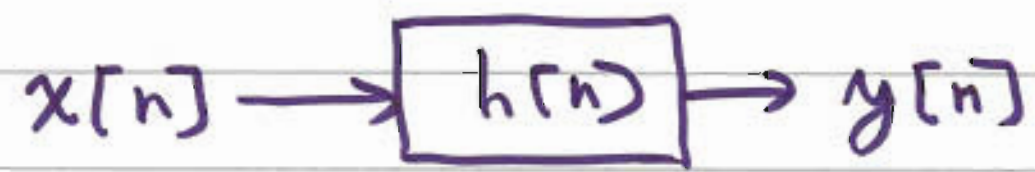


$$x[n] = -\sum_{k=1}^P a_k x[n-k] + v[n]$$

input: $v[n]$
output: $x[n]$

$$H(\omega) = H(z) \Big|_{z=e^{j\omega}} = \frac{1}{1 + \sum_{k=1}^P a_k e^{-j k \omega}}$$

• Input/Output Relationship for passing DT random process thru LTI filter



$r_{xx}[m] \xleftrightarrow{\text{DTFT}} S_{xx}(\omega)$

$r_{yy}[m] \xleftrightarrow{\text{DTFT}} S_{yy}(\omega)$

• Easy to prove: $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$

• where: $h[n] \xleftrightarrow{\text{DTFT}} H(\omega)$

• these relationships hold whether dealing with deterministic or stochastic auto correlation

$r_{yx}[m] = h[m] * r_{xx}[m]$

$r_{yy}[m] = r_{xx}[m] * r_{hh}[m]$

• Again, for AR process, input is $v(n)$ and output is $x(n)$

• Thus: $S_{xx}(\omega) = |H(\omega)|^2 S_{vv}(\omega)$

$$S_{vv}(\omega) = \text{DTFT}\{r_{vv}[n]\} = \text{DTFT}\{\sigma_w^2 \delta[n]\} = \sigma_w^2$$

$$H(\omega) = \frac{1}{1 + \sum_{k=1}^p a_k e^{-jk\omega}}$$

• Hence:

$$S_{xx}(\omega) = \frac{\sigma_w^2}{|1 + \sum_{k=1}^p a_k e^{-jk\omega}|^2} = S_{AR}(\omega)$$

• running white noise thru an generates this spectral density

all-pole filter

- $$x[n] = -\sum_{k=1}^P a_k x[n-k] + v[n]$$

- multiply both sides by $x^*[n-m]$ and take expected value

$$E\{x[n] x^*[n-m]\} = -\sum_{k=1}^P a_k E\{x[n-k] x^*[n-m]\} + E\{v[n] x^*[n-m]\}$$

$$r_{xx}[m] = -\sum_{k=1}^P a_k r_{xx}[m-k] + E\{v[n] x^*[n-m]\}$$

(for WSS $x[n]$: $E\{x[n-k] x^*[n-m]\} = r_{xx}[m-k]$)

- consider m to be strictly positive, since

$$r_{xx}[-m] = r_{xx}^*[m]$$

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$$\cdot x[n-m] = -\sum_{k=1}^p a_k x[n-m-k] + v[n-m]$$

• for $m > 0$: $x[n-m]$ depends only on past values of $v[n]$

$$\cdot \text{since } r_{vv}[m] = E\{v[n]v^*[n-m]\} = \sigma_w^2 \delta[m]$$

• it follows that for $m > 0$:

$$r_{xx}[m] = -\sum_{k=1}^p a_k r_{xx}[m-k] \quad m > 0$$

• Thus, we use $r_{xx}[m]$, $m=0, 1, \dots, p$ to set up a system of p equations in p unknowns

$$\begin{matrix}
 m=1 \\
 m=2 \\
 \vdots \\
 m=p
 \end{matrix}
 \begin{bmatrix}
 r_{xx}[1] \\
 r_{xx}[2] \\
 \vdots \\
 r_{xx}[p]
 \end{bmatrix}
 = -
 \begin{bmatrix}
 r_{xx}[0] & r_{xx}[-1] & \dots & r_{xx}[1-p] \\
 r_{xx}[1] & r_{xx}[0] & \dots & r_{xx}[2-p] \\
 \vdots & \vdots & \ddots & \vdots \\
 r_{xx}[p-1] & r_{xx}[p-2] & \dots & r_{xx}[0]
 \end{bmatrix}
 \begin{bmatrix}
 a_1 \\
 a_2 \\
 \vdots \\
 a_p
 \end{bmatrix}$$

$$\underline{r} = - \underline{R}_{xx} \underline{a} \quad \text{or} \quad \underline{R}_{xx} \underline{a} = - \underline{r}$$

$p \times p$ $p \times 1$ $p \times 1$

• using $r_{xx}[-m] = r_{xx}^*[m]$, we observe Hermitian (conjugate) symmetry about main diagonal as well as Toeplitz structure discussed previously

• for $m=0$ equation, we have

$$r_{xx}[0] = - \sum_{k=1}^p a_k r_{xx}[0-k] + E \{ v[n] x^*[n] \}$$

$$x[n] = - \sum_{k=1}^p x[n-k] + v[n]$$

only contain
 $v[n-1], \dots, v[n-p]$
past values of $v[n]$

• Thus:

$$r_{xx}[0] = - \sum_{k=1}^p a_k r_{xx}[k] + E \{ v[n] v^*[n] \}$$

$$r_{xx}[0] = - \sum_{k=1}^p a_k r_{xx}^*[k] + \sigma_w^2$$

$$\Rightarrow \sigma_w^2 = r_{xx}[0] + \sum_{k=1}^p a_k r_{xx}^*[k]$$