Parametric Spectral Estimation

(1)

- **ARMA Spectral Estimation**
- spectral density is modelled as

$$S_{ARMA}(\omega) = \sigma^2 \frac{\left| \sum_{k=0}^{M} b_k e^{-jk\omega} \right|^2}{\left| \sum_{k=0}^{N} a_k e^{-jk\omega} \right|^2}$$

rational spectrum

$a_0 = 1$

- Notation from statistics literature on ARMA processes $M = q$, $N = p$
- **Indirect spectral estimation:** compute b_k, $k=0,\ldots,q$ and a_k, $k=1,\ldots,p$ from the estimated autocorrelation values

- only need (theoretically) autocorrelations for lag values $m = 0, 1, \ldots, p+q$ to determine b_k, $k = 0, 1, \ldots, q$ and a_k, $k = 1, \ldots, p$

(note: $r_{xx}[-m] = r_{xx}^*[m]$)

- ARMA spectral estimation works well BUT the ARMA model parameters are nonlinearly related to the autocorrelation values
  - issue of computational complexity

- AR spectral estimation

(3)

$$S_{AR}(\omega) = \frac{\sigma^2}{\left|\sum_{k=0}^{p} a_k e^{-jk\omega}\right|^2} \qquad a_0 = 1$$

- AR model parameters are computed from the estimated/measured auto-correlation values via a <u>linear</u> system of equations of the form

$$\underset{p\times p}{\underline{R}_{xx}}\ \underset{p\times 1}{\underline{a}} = -\underset{p\times 1}{\underline{r}}$$

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$$

$\underline{R}_{xx}$ is a Toeplitz-Hermitian matrix which is formed from autocorrelations for lag values $0, 1, \ldots, p$ (4)

$$\begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] & r_{xx}^*[2] & \cdots & r_{xx}^*[p-1] \\ r_{xx}[1] & r_{xx}[0] & r_{xx}^*[1] & \cdots & r_{xx}^*[p-2] \\ r_{xx}[2] & r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}^*[p-3] \\ \vdots & & \ddots & \ddots & \\ r_{xx}[p-1] & r_{xx}[p-2] & & & r_{xx}[0] \end{bmatrix}$$

- Toeplitz: constant along any diagonal
- Hermitian: conjugate-symmetric about main diagonal

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- Due to Toeplitz structure of $\underline{R}_{xx}$ $\underline{R}_{xx}\underline{a} = -\underline{r}$ can be solved with low complexity $\Rightarrow$ in fact, there is no need to even form the matrix $\underline{R}_{xx}$
  - $\Rightarrow$ Levinson-Durbin algorithm

- Thus, AR spectral estimation is used more in practice than ARMA spectral estimation, even though ARMA generally performs better than AR
  - can increase the value of p to make AR perform almost as well as ARMA

- So, the steps are:

1. Estimate $r_{xx}[m]$ for lag values $m=0,1,\ldots,P$

$$\hat{r}_{xx}[m] = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x^*[n]\, x[n+m]$$

or N

$m=0,1,\ldots,P$

2. Solve $\underline{R}_{xx}\,\underline{a} = -\underline{r}$ for the AR model parameters, $\hat{a}_k$, $k=1,\ldots,P$ via the Levinson-Durbin algorithm

3. Plug the AR model parameters into the AR spectral estimate:

$$S_{xx}(\omega) = \frac{\sigma^2}{\left|1 + \sum_{k=1}^{P} a_k e^{-jk\omega}\right|^2}$$

(talk about σ^2 later)

- Relationship between AR model parameters a_k, $k=1,\ldots,p$ and autocorrelation values $r_{xx}[m]$, $m=0,1,\ldots,p$

- Observation: consider passing white noise $\nu[n]$ through an all-pole filter

$$\nu[n] \longrightarrow \boxed{H(z)=\frac{1}{1+\sum_{k=1}^{p} a_k z^{-k}}} \longrightarrow x[n]$$

$$r_{\nu\nu}[m]=\sigma_w^2\,\delta[m]$$

$$x[n]=-\sum_{k=1}^{p} a_k\, x[n-k]+\nu[n] \quad \left\}\begin{array}{l}\text{input: } \nu(n)\\ \text{output: } x(n)\end{array}\right.$$

$$H(\omega)=H(z)\Big|_{z=e^{j\omega}}=\frac{1}{1+\sum_{k=1}^{p} a_k e^{-jk\omega}}$$

- Input / Output Relationship for passing DT random process thru LTI filter

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$r_{xx}[m] \xleftrightarrow{\text{DTFT}} S_{xx}(\omega) \qquad r_{yy}[m] \xleftrightarrow{\text{DTFT}} S_{yy}(\omega)$$

- Easy to prove: $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$
  - where: $h[n] \xleftrightarrow{\text{DTFT}} H(\omega)$

- these relationships hold whether dealing with deterministic <u>or</u> stochastic autocorrelation

$$r_{yx}[m] = h[m] * r_{xx}[m]$$

$$r_{yy}[m] = r_{xx}[m] * r_{hh}[m]$$

- Again, for AR process, input is $\nu[n]$ and output is $x[n]$

- <u>Thus:</u> $S_{xx}(\omega) = |H(\omega)|^2 S_{\nu\nu}(\omega)$

$$S_{\nu\nu}(\omega) = \text{DTFT}\{r_{\nu\nu}[m]\} = \text{DTFT}\{\sigma_w^2 \delta[m]\} = \sigma_w^2$$

$$H(\omega) = \frac{1}{1+\sum_{k=1}^{p} a_k e^{-jk\omega}}$$

- <u>Hence:</u>

$$S_{xx}(\omega) = \frac{\sigma_w^2}{\left|1+\sum_{k=1}^{p} a_k e^{-j\omega}\right|^2} = S_{AR}(\omega)$$

- running white noise thru an all-pole filter generates this spectral density

- $$x[n] = -\sum_{k=1}^{p} a_k x[n-k] + \nu[n]$$

- multiply both sides by $x^*[n-m]$ and take expected value

$$E\{x[n]x^*[n-m]\} = -\sum_{k=1}^{p} a_k E\{x[n-k]x^*[n-m]\}$$
$$+ E\{\nu[n]x^*[n-m]\}$$

$$r_{xx}[m] = -\sum_{k=1}^{p} a_k r_{xx}[m-k] + E\{\nu[n]x^*[n-m]\}$$

(for WSS $x[n]$: $E\{x[n-k]x^*[n-m]\} = r_{xx}[m-k]$)

- Consider m to be strictly positive, since

$$r_{xx}[-m] = r_{xx}^*[m]$$

- $$x[n-m] = -\sum_{k=1}^{p} a_k x[n-m-k] + \nu[n-m]$$

- for $m>0$: $x[n-m]$ depends only on past values of $\nu[n]$

- since $r_{\nu\nu}[m] = E\{\nu[n]\nu^*[n-m]\} = \sigma_w^2 \delta[m]$

- it follows that for $m>0$:

$$r_{xx}[m] = -\sum_{k=1}^{p} a_k r_{xx}[m-k] \qquad m>0$$

- Thus, we use $r_{xx}[m]$, $m=0,1,\ldots,p$ to set up a system of p equations in p unknowns

$$\begin{array}{c} m=1 \\ m=2 \\ \\ \\ m=p \end{array} \begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \\ \vdots \\ r_{xx}[p] \end{bmatrix} = - \begin{bmatrix} r_{xx}[0] & r_{xx}[-1] & \cdots & r_{xx}[1-p] \\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[2-p] \\ \vdots & \vdots & & \vdots \\ r_{xx}[p-1] & r_{xx}[p-2] & \cdots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$$

$$\underline{r} = -\underline{R}_{xx}\,\underline{a} \quad \text{or} \quad \underset{p\times p}{\underline{R}_{xx}}\;\underset{p\times 1}{\underline{a}} = -\underset{p\times 1}{\underline{r}}$$

- using $r_{xx}[-m] = r^*_{xx}[m]$, we observe Hermitian (conjugate) symmetry about main diagonal as well as Toeplitz structure disussed previously

- for $m=0$ equation, we have

$$r_{xx}[0] = -\sum_{k=1}^{p} a_k r_{xx}[0-k] + E\{\nu[n]\, \underbrace{x^*[n]}\}$$

$$x[n] = -\underbrace{\sum_{k=1}^{p} x[n-k]}_{\substack{\text{only contain} \\ \nu[n-1], \ldots, \nu[n-p] \\ \text{past values of } \nu[n]}} + \nu[n]$$

- Thus:

$$r_{xx}[0] = -\sum_{k=1}^{p} a_k r_{xx}[-k] + E\{\nu[n]\, \nu^*[n]\}$$

$$r_{xx}[0] = -\sum_{k=1}^{p} a_k r_{xx}^*[k] + \sigma_w^2$$

$$\Rightarrow \boxed{\sigma_w^2 = r_{xx}[0] + \sum_{k=1}^{p} a_k r_{xx}^*[k]}$$