• PRFB Notes: Part 2

• With synthesis filters $g_m[n] = h_m^*[-n]$

$m = 0, 1, \ldots, M-1$

the conditions for PR are:

$$\sum_{m=0}^{M-1} h_m[n] * h_m^*[-n] = c\,\delta[n] \quad \text{(Distortionless)}$$

$$\left.\begin{aligned}
&\sum_{m=0}^{M-1} \left\{e^{j\frac{2\pi}{M}n} h_m[n]\right\} * h_m^*[-n] = 0 \quad \forall n \\
&\sum_{m=0}^{M-1} \left\{e^{j\frac{4\pi}{M}n} h_m[n]\right\} * h_m^*[-n] = 0 \quad \forall n \\
&\qquad \vdots \\
&\sum_{m=0}^{M-1} \left\{e^{j(M-1)\frac{2\pi}{M}n} h_m[n]\right\} * h_m^*[-n] = 0 \quad \forall n
\end{aligned}\right\} \text{Alias-Free}$$

- Consider FIR analysis filters of length M
- let the values of the filter be in a column vector

$M \times 1$

$$\underline{h}_m = \begin{bmatrix} h_m[0] \\ h_m[1] \\ \vdots \\ h_m[M-1] \end{bmatrix} \qquad m = 0, 1, \ldots, M-1$$

- Also, define: $$\underline{f}(\omega) = \begin{bmatrix} 1 \\ e^{j\omega} \\ e^{j2\omega} \\ \vdots \\ e^{j(M-1)\omega} \end{bmatrix} \qquad M \times 1$$

- Then:

$$H_m(\omega) = \underline{f}^H(\omega)\,\underline{h}_m$$

($H \equiv *T$ = conjugate-transpose)

superscript

$$H_m^*(\omega) = \text{DTFT}\{h_m^*[-n]\} = \underline{h}_m^H \underline{f}(\omega) = \underline{h}_m^H \underline{f}(\omega)$$

$$H_m(\omega)\,H_m^*(\omega) = \underline{f}^H(\omega)\,\underline{h}_m\,\underline{h}_m^H\,f(\omega)$$

• For Distortionless, we require:

$$\sum_{m=0}^{M-1} H_m(\omega)\, H_m^*(\omega) = C$$

$$= \sum_{m=0}^{M-1} \underline{f}^H(\omega)\, \underline{h}_m \underline{h}_m^H\, \underline{f}(\omega)$$

$$= \underline{f}^H(\omega) \sum_{m=0}^{M-1} \underline{h}_m \underline{h}_m^H\, \underline{f}(\omega)$$

<u>Note</u>: if filter vectors are mutually orthogonal, with the same energy, E_h, then:

$$\sum_{m=0}^{M-1} \underline{h}_m \underline{h}_m^H = E_h \underbrace{\underline{I}_M}_{M \times M}$$ Identity Matrix

Then:

$$\sum_{m=0}^{M-1} H_m(\omega) H_m^*(\omega) = \underline{f}^H(\omega)\, E_h\, \underline{I}_M\, \underline{f}(\omega) = M E_h \checkmark$$

Distortionless

Check Alias-Free:

$$\sum_{m=0}^{M-1} \underline{f}^H(\omega) \underbrace{\begin{bmatrix} 1 & & & \\ & e^{j k \frac{2\pi}{M}} & & 0 \\ 0 & & e^{j 2k \frac{2\pi}{M}} & \\ & & & \ddots \\ & & & & e^{j (M-1) k \frac{2\pi}{M}} \end{bmatrix}}_{M \times M \text{ Diagonal Matrix}} \underline{h}_m \underline{h}_m^H \underline{f}(\omega) = 0 \quad \forall \omega$$

for $k = 1, 2, \ldots, M-1$

$$= \underline{f}^H(\omega) \begin{bmatrix} 1 & & & \\ & e^{j k \frac{2\pi}{M}} & & 0 \\ & & \ddots & \\ 0 & & & e^{j (M-1) k \frac{2\pi}{M}} \end{bmatrix} \underbrace{\sum_{m=0}^{M-1} \underline{h}_m \underline{h}_m^H}_{E_h \underline{I}_M} \underline{f}(\omega)$$

$$= \left[1 \;\; e^{-j\omega} \;\; e^{-j2\omega} \cdots e^{-j(M-1)\omega}\right] \begin{bmatrix} 1 \\ e^{j\frac{2\pi}{M}k} e^{j\omega} \\ e^{j2\frac{2\pi}{M}k} e^{j2\omega} \\ \vdots \\ e^{j(M-1)k\frac{2\pi}{M}} e^{j(M-1)\omega} \end{bmatrix}$$

$$= \sum_{m=0}^{M-1} e^{j\frac{2\pi}{M}km} \qquad \text{for } k = 1, 2, \ldots, M-1$$

$$= \frac{1 - e^{jk\frac{2\pi}{M}M}}{1 - e^{jk\frac{2\pi}{M}}} = \frac{1-1}{1-e^{jk\frac{2\pi}{M}}} = 0$$

- Alias-Free holds!

- An $M \times M$ M-pt. DFT matrix
  - each row is a sinewave of length M
  - the frequencies of the sinewaves are:

$$0, \frac{2\pi}{M}, \frac{4\pi}{M}, \ldots, (M-1)\frac{2\pi}{M}$$

  - the sinewaves are periodic with period M
    - most of them repeat more quickly
  - the sinewaves are mutually orthogonal with equal energy (= M per sinewave)
  - THUS, they can be used as bandpass filters for Perfect Reconstruction