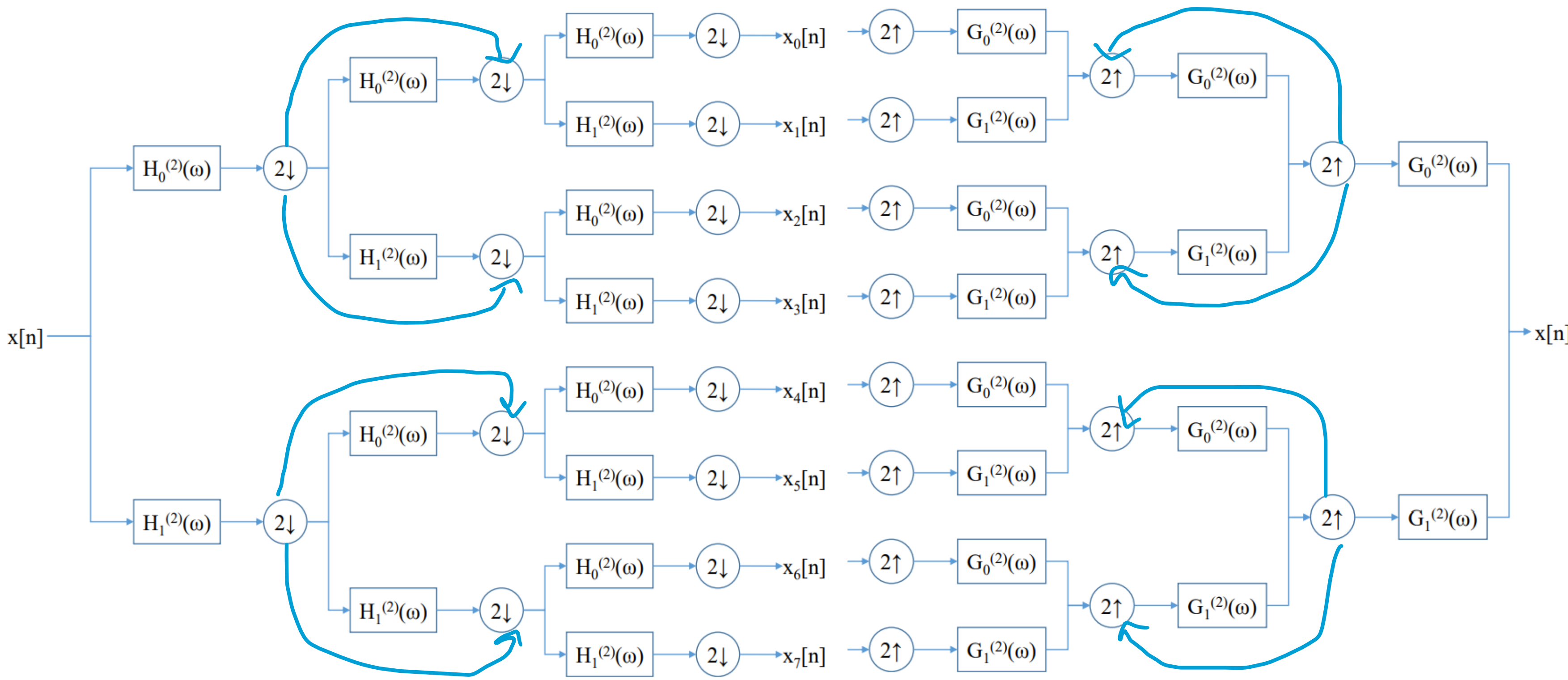
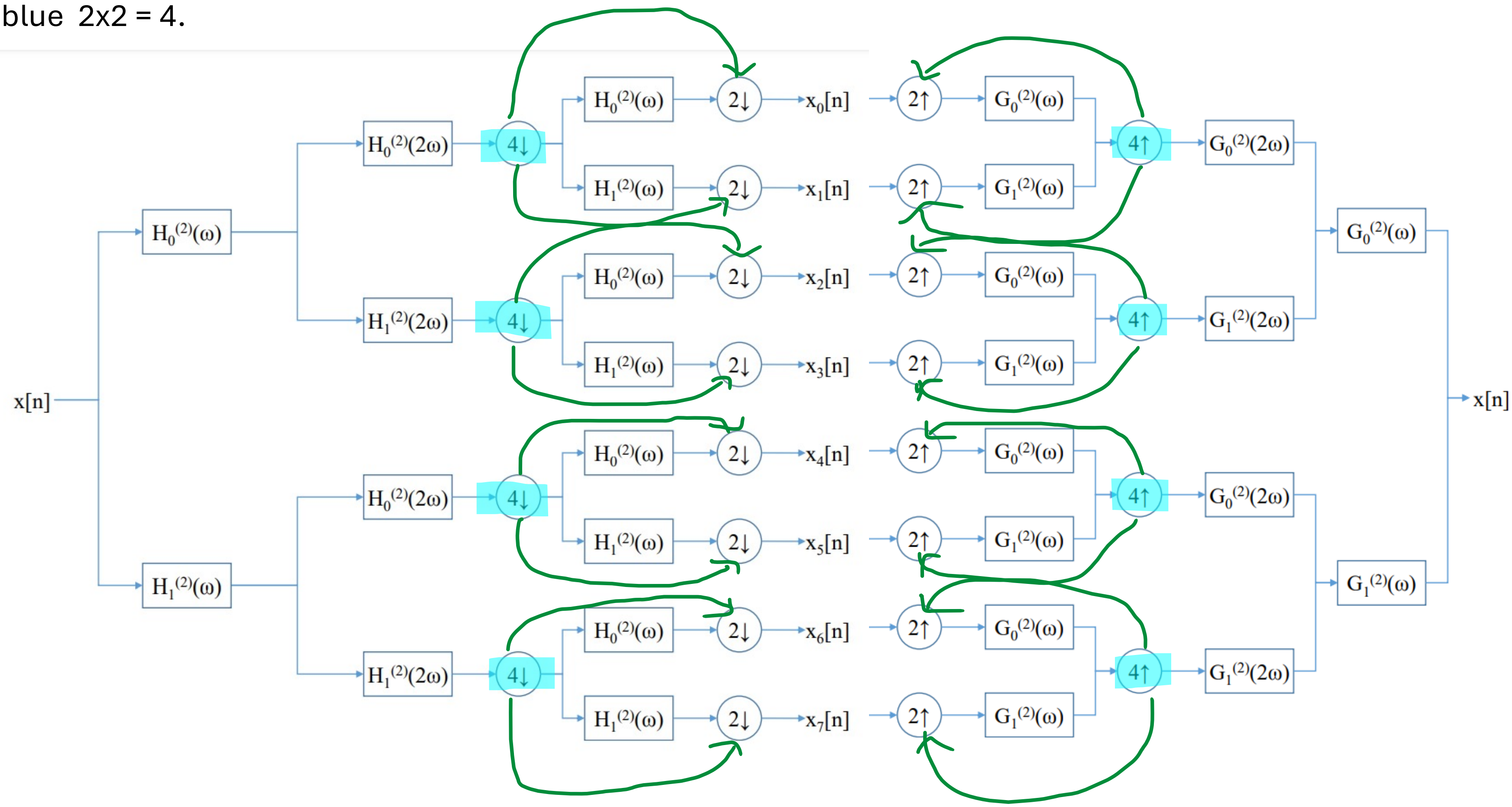
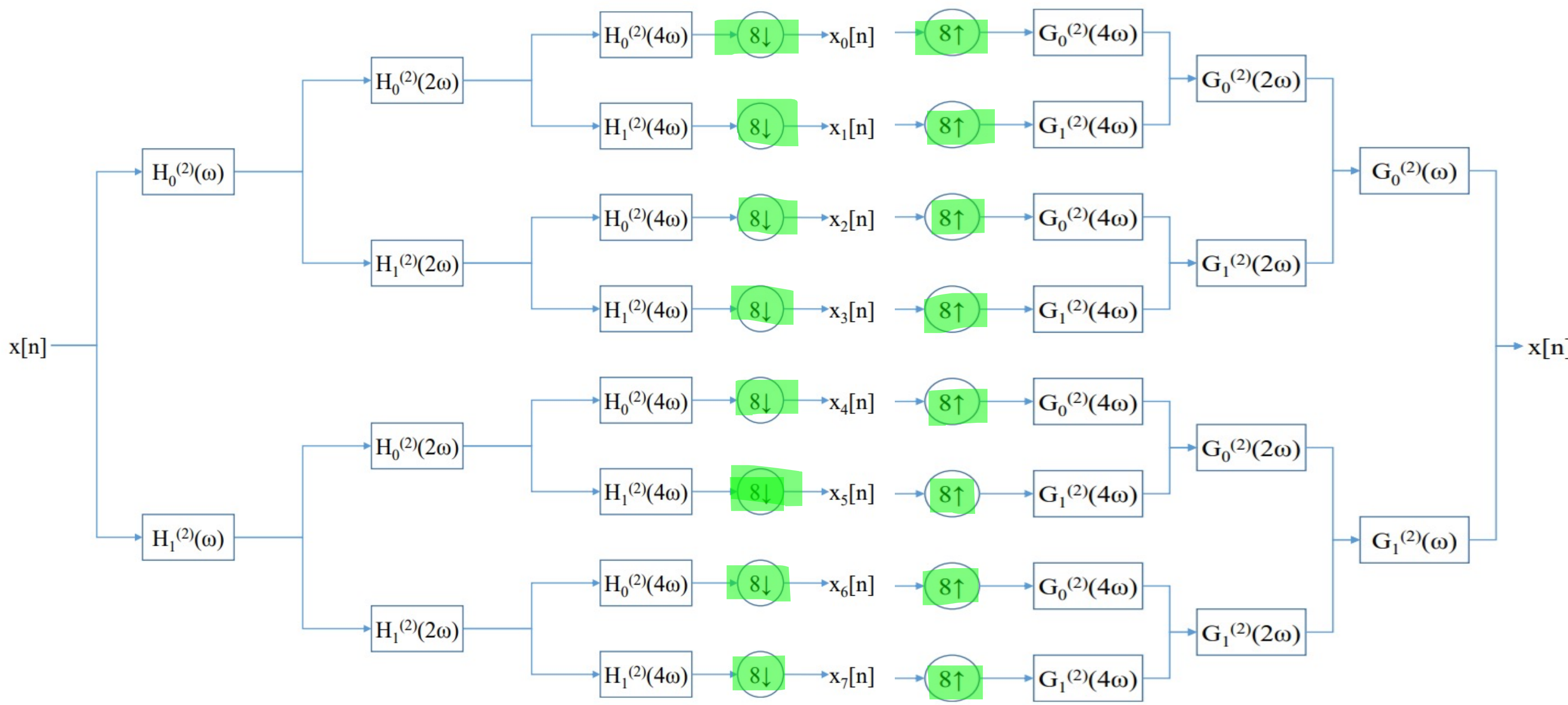
Using the Noble identity's Proofs, it allows us to move the down sampling and up sampling it will be a **product** of the decimation factors / the up-sampling factors

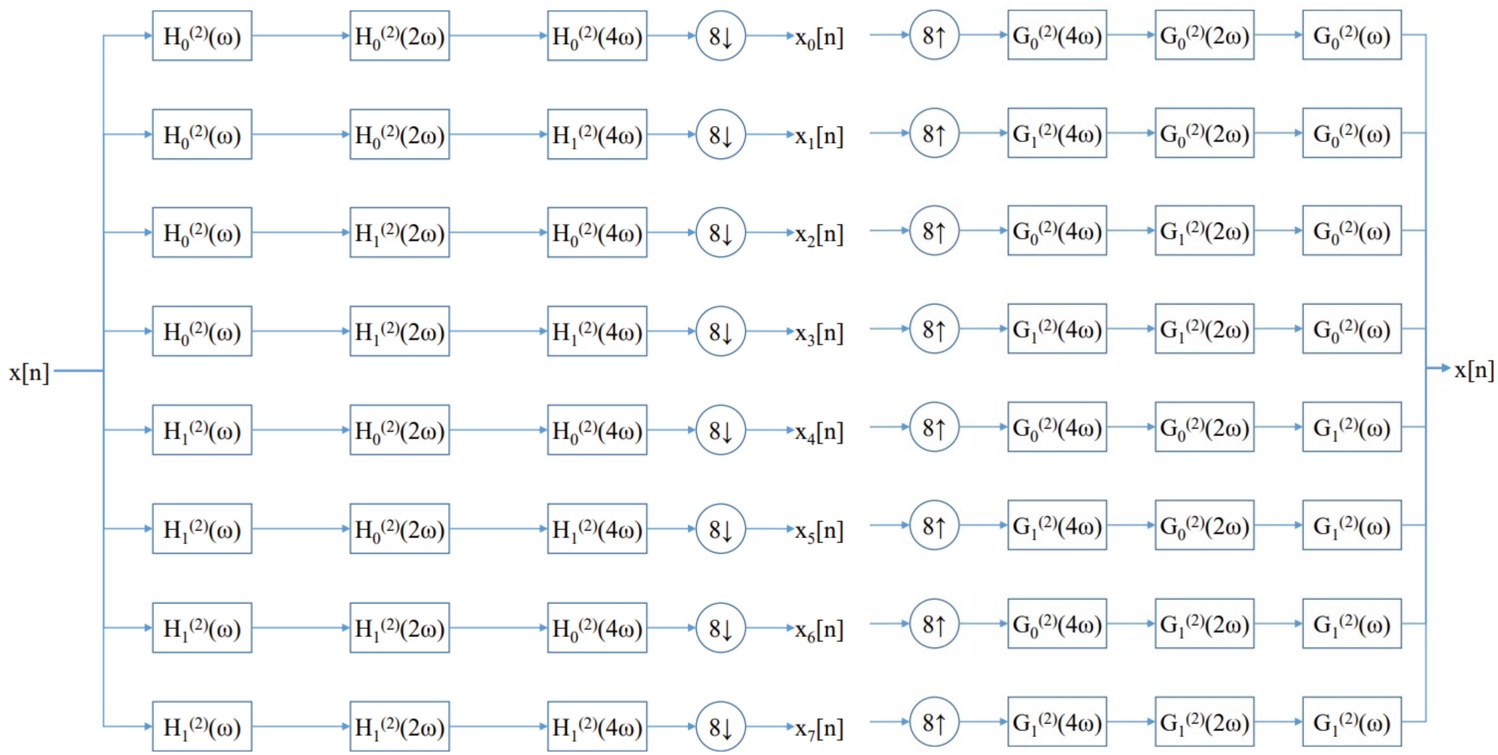


Effects of Blue arrow are highlighted
in blue 2x2 = 4.



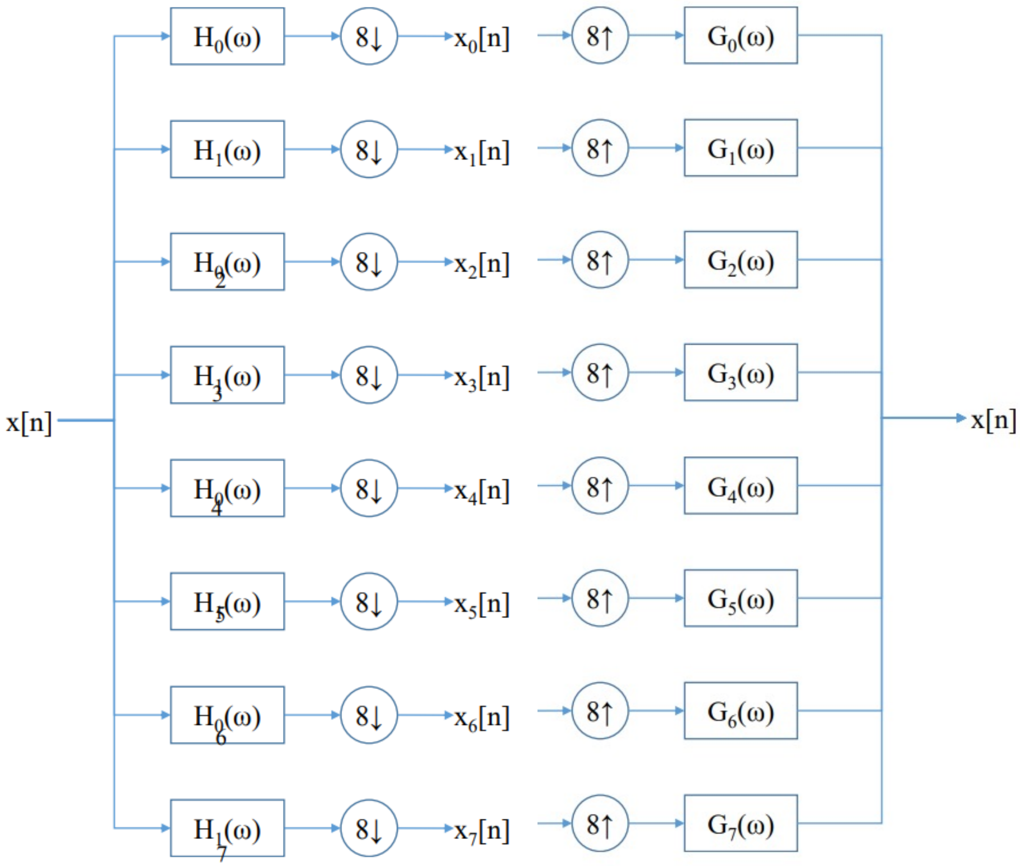
Effects of Green arrow are highlighted
in green 4x2 = 8.





x[n]
$H_0^{(2)}(\omega)$ $H_0^{(2)}(2\omega)$ $H_0^{(2)}(4\omega)$ 8↓ $x_0[n]$ 8↑ $G_0^{(2)}(4\omega)$ $G_0^{(2)}(2\omega)$ $G_0^{(2)}(\omega)$
$H_0^{(2)}(\omega)$ $H_0^{(2)}(2\omega)$ $H_1^{(2)}(4\omega)$ 8↓ $x_1[n]$ 8↑ $G_1^{(2)}(4\omega)$ $G_0^{(2)}(2\omega)$ $G_0^{(2)}(\omega)$
$H_0^{(2)}(\omega)$ $H_1^{(2)}(2\omega)$ $H_0^{(2)}(4\omega)$ 8↓ $x_2[n]$ 8↑ $G_0^{(2)}(4\omega)$ $G_1^{(2)}(2\omega)$ $G_0^{(2)}(\omega)$
$H_0^{(2)}(\omega)$ $H_1^{(2)}(2\omega)$ $H_1^{(2)}(4\omega)$ 8↓ $x_3[n]$ 8↑ $G_1^{(2)}(4\omega)$ $G_1^{(2)}(2\omega)$ $G_0^{(2)}(\omega)$
$H_1^{(2)}(\omega)$ $H_0^{(2)}(2\omega)$ $H_0^{(2)}(4\omega)$ 8↓ $x_4[n]$ 8↑ $G_0^{(2)}(4\omega)$ $G_0^{(2)}(2\omega)$ $G_1^{(2)}(\omega)$
$H_1^{(2)}(\omega)$ $H_0^{(2)}(2\omega)$ $H_1^{(2)}(4\omega)$ 8↓ $x_5[n]$ 8↑ $G_1^{(2)}(4\omega)$ $G_0^{(2)}(2\omega)$ $G_1^{(2)}(\omega)$
$H_1^{(2)}(\omega)$ $H_1^{(2)}(2\omega)$ $H_0^{(2)}(4\omega)$ 8↓ $x_6[n]$ 8↑ $G_0^{(2)}(4\omega)$ $G_1^{(2)}(2\omega)$ $G_1^{(2)}(\omega)$
$H_1^{(2)}(\omega)$ $H_1^{(2)}(2\omega)$ $H_1^{(2)}(4\omega)$ 8↓ $x_7[n]$ 8↑ $G_1^{(2)}(4\omega)$ $G_1^{(2)}(2\omega)$ $G_1^{(2)}(\omega)$
x[n]

Because of the Noble Identity if F(w) = H(MW) then systems must have the same I/O which allows the condensation of the tree in this step.



$$
\begin{aligned}
h_0[n] &= h_0^{(2)}[n] * h_{0\uparrow 2}^{(2)}[n] * h_{0\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad H_0(\omega) &= H_0^{(2)}(\omega) \cdot H_0^{(2)}(2\omega) \cdot H_0^{(2)}(4\omega) \\
h_1[n] &= h_0^{(2)}[n] * h_{0\uparrow 2}^{(2)}[n] * h_{1\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad H_1(\omega) &= H_0^{(2)}(\omega) \cdot H_0^{(2)}(2\omega) \cdot H_1^{(2)}(4\omega) \\
h_2[n] &= h_0^{(2)}[n] * h_{1\uparrow 2}^{(2)}[n] * h_{0\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad H_2(\omega) &= H_0^{(2)}(\omega) \cdot H_1^{(2)}(2\omega) \cdot H_0^{(2)}(4\omega) \\
h_3[n] &= h_0^{(2)}[n] * h_{1\uparrow 2}^{(2)}[n] * h_{1\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad H_3(\omega) &= H_0^{(2)}(\omega) \cdot H_1^{(2)}(2\omega) \cdot H_1^{(2)}(4\omega) \\
h_4[n] &= h_1^{(2)}[n] * h_{0\uparrow 2}^{(2)}[n] * h_{0\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad H_4(\omega) &= H_1^{(2)}(\omega) \cdot H_0^{(2)}(2\omega) \cdot H_0^{(2)}(4\omega) \\
h_5[n] &= h_1^{(2)}[n] * h_{0\uparrow 2}^{(2)}[n] * h_{1\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad H_5(\omega) &= H_1^{(2)}(\omega) \cdot H_0^{(2)}(2\omega) \cdot H_1^{(2)}(4\omega) \\
h_6[n] &= h_1^{(2)}[n] * h_{1\uparrow 2}^{(2)}[n] * h_{0\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad H_6(\omega) &= H_1^{(2)}(\omega) \cdot H_1^{(2)}(2\omega) \cdot H_0^{(2)}(4\omega) \\
h_7[n] &= h_1^{(2)}[n] * h_{1\uparrow 2}^{(2)}[n] * h_{1\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad H_7(\omega) &= H_1^{(2)}(\omega) \cdot H_1^{(2)}(2\omega) \cdot H_1^{(2)}(4\omega)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
g_0[n] &= g_0^{(2)}[n] * g_{0\uparrow 2}^{(2)}[n] * g_{0\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad G_0(\omega) &= G_0^{(2)}(\omega) \cdot G_0^{(2)}(2\omega) \cdot G_0^{(2)}(4\omega) \\
g_1[n] &= g_0^{(2)}[n] * g_{0\uparrow 2}^{(2)}[n] * g_{1\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad G_1(\omega) &= G_0^{(2)}(\omega) \cdot G_0^{(2)}(2\omega) \cdot G_1^{(2)}(4\omega) \\
g_2[n] &= g_0^{(2)}[n] * g_{1\uparrow 2}^{(2)}[n] * g_{0\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad G_2(\omega) &= G_0^{(2)}(\omega) \cdot G_1^{(2)}(2\omega) \cdot G_0^{(2)}(4\omega) \\
g_3[n] &= g_0^{(2)}[n] * g_{1\uparrow 2}^{(2)}[n] * g_{1\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad G_3(\omega) &= G_0^{(2)}(\omega) \cdot G_1^{(2)}(2\omega) \cdot G_1^{(2)}(4\omega) \\
g_4[n] &= g_1^{(2)}[n] * g_{0\uparrow 2}^{(2)}[n] * g_{0\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad G_4(\omega) &= G_1^{(2)}(\omega) \cdot G_0^{(2)}(2\omega) \cdot G_0^{(2)}(4\omega) \\
g_5[n] &= g_1^{(2)}[n] * g_{0\uparrow 2}^{(2)}[n] * g_{1\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad G_5(\omega) &= G_1^{(2)}(\omega) \cdot G_0^{(2)}(2\omega) \cdot G_1^{(2)}(4\omega) \\
g_6[n] &= g_1^{(2)}[n] * g_{1\uparrow 2}^{(2)}[n] * g_{0\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad G_6(\omega) &= G_1^{(2)}(\omega) \cdot G_1^{(2)}(2\omega) \cdot G_0^{(2)}(4\omega) \\
g_7[n] &= g_1^{(2)}[n] * g_{1\uparrow 2}^{(2)}[n] * g_{1\uparrow 4}^{(2)}[n] & \xrightarrow{\mathcal{DTFT}} \quad G_7(\omega) &= G_1^{(2)}(\omega) \cdot G_1^{(2)}(2\omega) \cdot G_1^{(2)}(4\omega)
\end{aligned}
\tag{2}
$$

note: $g_0^{(2)}[n] = h_0^{(2)}[n]$, & $g_1^{(2)}[n] = -h_1^{(2)}[n]$