OFDM Notes

Recall Synchronous CDMA Downlink Using Orthogonal Codes

- For one CDMA frame/block:

$$x[n] = \sum_{k=0}^{N-1} b_k \, c_k[n]$$

where $c_k[n]$ is a code of length N

$k = 0, 1, \ldots, N-1$ code-index

$n = 0, 1, \ldots, N-1$ time-index

- Use orthogonal codes

e.g., Walsh-Hadamard codes:

$$\sum_{n=0}^{N-1} c_k[n] \, c_l^*[n] = N \, \delta[k-l]$$

i.e., $= 0$ if $k \neq l$

- Because the codes are orthogonal, (2)
  the problem of determining the information bit b_k is simplified, essentially decoupled

$$\frac{1}{N}\sum_{n=0}^{N-1} x[n]\, c_k^*[n] = b_k \qquad k=0,1,\dots,N-1$$

- Wireless communications is complicated by multipath propagation => reflections off buildings and other structures that arrive at different delays with different path gains
- effect of multipath is characterized or modeled as convolution with an FIR filter for short epochs of time

- Let $h[n]$ be the channel filter of length L characterizing the multipath (3)

  $\Rightarrow h[n] \neq 0$ only for $n = 0, 1, \ldots, L-1$

- Ignoring noise, we modeled the received signal as:

$$y[n] = x[n] * h[n]$$

$$= \left\{ \sum_{k=0}^{N-1} b_k \, c_k[n] \right\} * h[n]$$

$$= \sum_{k=0}^{N-1} b_k \left\{ c_k[n] * h[n] \right\}$$

- each $c_k[n]$ effectively replaced by

  $\Rightarrow c_k[n] * h[n]$ of length $N+L-1$

• Problems :

1. $\{c_k[n] * h[n]\}$ of length $N+L-1$ (4)

are not mutually orthogonal

2. The m-th CDMA block now spills into the (m+1)-th CDMA block

• Both problems increase the background level of multiuser access interference and degrade performance significantly

• OFDM can solve both problems

Orthogonal Frequency Division Multiplexing

- OFDM ⇒ one OFDM block:

$$x[n] = \sum_{k=0}^{N-1} b_k \, s_k[n]$$

- where $s_k[n]$ are DT sinewaves:

$$s_k[n] = e^{j\left(2\pi \frac{k}{N}\right) n}$$

time-index: $n = 0, 1, \ldots, N-1$

$$k = 0, 1, \ldots, N-1 \quad \Big\} \text{ sinewave frequency index}$$

- Like W-H codes in CDMA, the sinewaves are mutually orthogonal

$$\frac{1}{N} \sum_{n=0}^{N-1} s_k[n] \, s_\ell^*[n] = \delta[k-\ell]$$

$$= 0 \text{ if } k \neq \ell$$

Proof: $\displaystyle\sum_{n=0}^{N-1} s_k[n]\, s_l^*[n] =$

$$= \sum_{n=0}^{N-1} e^{j 2\pi \frac{k}{N} n}\, e^{-j 2\pi \frac{l}{N} n}$$

$$= \sum_{n=0}^{N-1} \left(e^{j 2\pi \frac{(k-l)}{N}} \right)^n$$

$$= \frac{1 - e^{j 2\pi \frac{(k-l)}{\cancel{N}} \cancel{N}}}{1 - e^{j 2\pi \frac{k-l}{N}}} = 0 \text{ if } k \neq l$$

• since $e^{j 2\pi (k-l)} = (1)^{k-l} = 1$

$k = 0, 1, \ldots, N-1$; $l = 0, 1, \ldots, N-1$

7

- With multipath, we have:

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=0}^{N-1} b_k \left\{ s_k[n] * h[n] \right\}$$

There are at least 2 ways of counteracting the effects of the multipath

1. Cyclic Prefix (CP) insertion
2. Zero Post-fix with overlap-add

Method 1 is most used in practice by far

- The convolution of a complex sinewave input of length M WITH a channel of length L (FIR filter of length L) yields:

- partial overlap at beginning:

  $L-1$ "transient" values $\Rightarrow$ <u>not</u> a sinewave

- full overlap in middle:

  $M + L - 1 - 2(L-1) = M - L + 1$ output values $\Rightarrow$ sinewave!

- partial overlap at end:

  $L-1$ "transient" values $\Rightarrow$ <u>not</u> a sinewave

- Throw away / discard transient points at beginning and end (assume $N \gg L$)

- Now, the N sinewaves are only orthogonal if they're of length N
- SO: we transmit longer sinewaves $M > N$ such that when we discard transient pts. at beginning and end, the remaining sinewaves are of length N
- length of full overlap region: $M - L + 1$

$$\Rightarrow \quad M - L + 1 = N$$

$$\Rightarrow \quad M = N + L - 1$$

- In practice, we send sinewaves of length $M = N + L$ and discard L pts. at beginning and $L-1$ pts. at end

- OFDM $\Rightarrow$ one block:

$$x[n] = \sum_{k=0}^{N-1} b_k \, \tilde{s}_k[n]$$

$$\tilde{s}_k[n] = e^{j\left(2\pi \frac{k}{N}\right)n} \left\{ u[n+L] - u[n-N] \right\}$$

$n = -L, \ldots, -1, 0, 1, \ldots, N-1$

$\Rightarrow$ length $N+L$

- Since distributive property applies

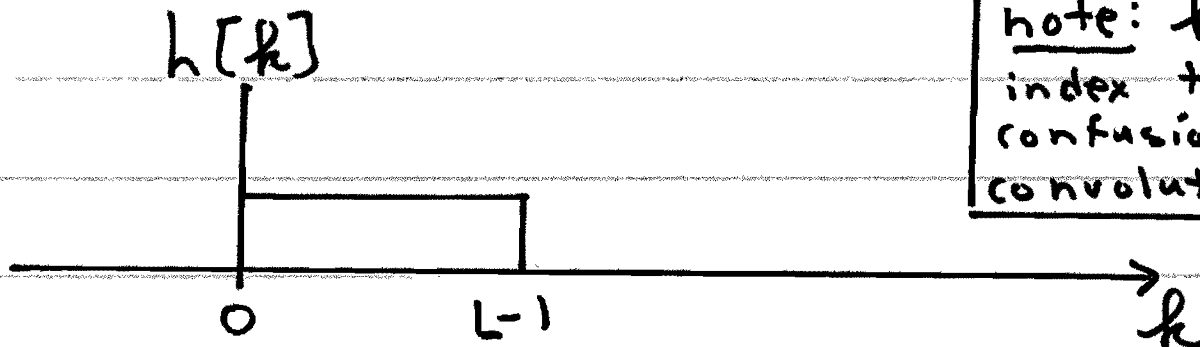
$$x[n] * h[n] = \sum_{k=0}^{N-1} b_k \left\{ \tilde{s}_k[n] * h[n] \right\}$$

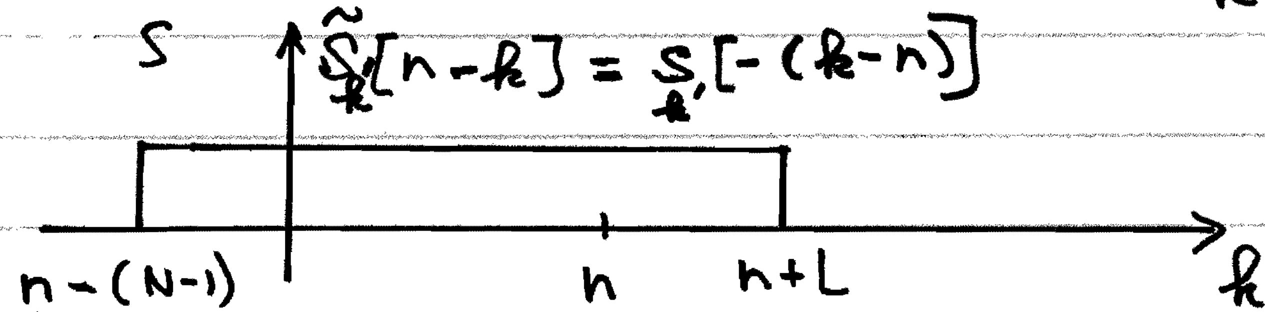
- examine convolution of $\tilde{s}_k[n]$ with $h[n]$ which is only nonzero for $n = 0, 1, \ldots, L-1$

$$y_{k'}[n] = \tilde{s}_{k'}[n] * h[n]$$

- In the graphical method of convolution, I will flip around the sinewave:

note: k' sinewave index to avoid confusion with k in convolution summation





- the "box shape" is only to show the nonzero region to determine limits on convolution summation

Limits on Convolution Sum:

• Partial overlap:

$$\left.\begin{array}{l} n+L \geq 0 \\ n+L < L-1 \end{array}\right\} -L \leq n < -1 \text{ (discard)} \quad \Big\rangle \sum_{k=0}^{n+L}$$

• Full overlap:

$$\left.\begin{array}{l} n+L \geq L-1 \\ n-(N-1) \leq 0 \end{array}\right\} -1 \leq n \leq N-1 \quad \left(\begin{array}{l}\text{we'll discard} \\ n=-1 \text{ pt.}\end{array}\right) \quad \Big\rangle \sum_{k=0}^{L-1}$$

• Partial overlap:

$$\left.\begin{array}{l} n-(N-1) \leq L-1 \\ n-(N-1) > 0 \end{array}\right\} N-1 < n \leq N+L-2 \text{ (discard)} \quad \Big\rangle \sum_{k=n-(N-1)}^{L-1}$$

- **Partial overlap:** $-L \le n < -1$

$$y_{k'}[n] = \sum_{k=0}^{n+L} h[k]\, e^{j\left(2\pi \frac{k'}{N}\right)(n-k)}$$

$$= \underbrace{\left\{ \sum_{k=0}^{n+L} h[k]\, e^{-j\left(2\pi \frac{k'}{N}\right) k} \right\}}_{\text{amplitude changing with } n} e^{j\left(2\pi \frac{k'}{N}\right) n}$$

$\Rightarrow$ not a sinewave

$\Rightarrow$ discard $L-1$ "transient" points

- Full Overlap: $-1 \le n \le N-1$

$$y_{k'}[n] = \sum_{k=0}^{L-1} h[k]\, e^{j\left(\frac{2\pi k'}{N}\right)(n-k)}$$

$$= \underbrace{\left\{\sum_{k=0}^{L-1} h[k]\, e^{-j\left(\frac{2\pi k'}{N}\right)k}\right\}}_{\text{doesn't depend on } n \Rightarrow \text{ sinewave!}} e^{j\left(\frac{2\pi k'}{N}\right)n}$$

$$H(\omega) = \sum_{n=0}^{L-1} h[n]\, e^{-j\omega n} \quad \Big\} \text{ DTFT of } h(n)$$

Define: $H_N(k) = H(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$

Then:

$$y_{k'}[n] = H_N(k')\, e^{j 2\pi \frac{k'}{N} n}$$

- Thus, full-overlap region mimicks eigen-function phenomena with complex sinewave turned-on forever into LTI system

$$e^{j\omega_0 n} \xrightarrow{\boxed{h[n]}} H(\omega_0)\, e^{j\omega_0 n}$$

- because in full-overlap region, filter can't distinguish between finite-length sinewave and infinite-length sinewave
- again, for full-overlap region:

$$y_{k'}[n] = H_N(k')\, e^{j\left(2\pi \frac{k'}{N}\right) n} \qquad -1 \le n \le N-1$$

- but desire sinewaves of length N for orthogonality $\Rightarrow$ throw away $n=-1$ point

- Partial overlap at end:

$$N-1 < n \leq N+L-2$$

$$y_{k'}[n] = \sum_{k=n-(N-1)}^{L-1} h[k]\, e^{j\left(2\pi \frac{k'}{N}\right)(n-k)}$$

$$= \underbrace{\left\{ \sum_{k=n-(N-1)}^{L-1} h[k]\, e^{-j\left(2\pi \frac{k'}{N}\right)k} \right\}}_{\text{amplitude depends on } n} e^{j\,2\pi \frac{k'}{N} n}$$

$\Rightarrow$ not a sinewave

$\Rightarrow$ discard $L-1$ "transient" points at the end

• Invoking linearity / superposition:

$$x[n] = \sum_{k=0}^{N-1} b_k \, e^{j\left(2\pi \frac{k}{N}\right)n} \left\{ u[n+L] - u[n-N] \right\}$$

$$\underbrace{\tilde{y}[n] = \underset{\text{length } N+L}{x[n]} * \underset{\text{length } L}{h[n]}}_{\text{convolution length: } L + N + L - 1}$$

convolution length: $N + L + L - 1$

convolution length: $\underset{\text{discard}}{L} + \underset{\text{keep}}{N} + \underset{\text{discard}}{L-1}$

$$y[n] = \tilde{y}[n] \cdot \left\{ u[n] - u[n-N] \right\}$$

$$= \sum_{k=0}^{N-1} b_k \, H_N(k) \, e^{j\left(2\pi \frac{k}{N}\right)n} \left(u[n] - u[n-N] \right)$$

THUS:

$$\frac{1}{N}\sum_{n=0}^{N-1} y[n]\, s_k^*[n] = \frac{1}{N}\sum_{n=0}^{N-1} y[n]\, e^{-j2\pi\frac{k}{N}n}$$

$$= H_N(k)\, b_k \qquad k = 0, 1, \ldots, N-1$$

=> because sinewaves are orthogonal

- We will show later how to estimate $H_N(k)$ so we can then determine b_k
- Equation at top is an N-pt. DFT
  => to be covered later in Chap 7
- which can be efficiently computed when N is a power of two => radix-2 FFT
  => Chap 8

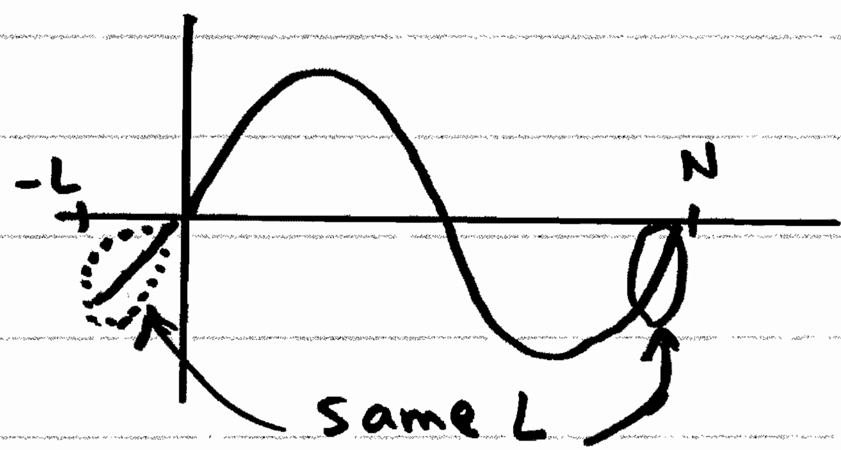
• THAT IS: after we discard L pts. at the beginning and L-1 pts. at end, the N-pt DFT (computed via an FFT) is the mechanism by which we multiply the received signal (truncated) by the complex-conjugate of each sine-wave and sum FOR ALL OF THE SINEWAVES ⇒ KIND OF COLLECTIVELY IN PARALLEL

• DON'T THINK OF IT AS TRANSFORMING FROM THE TIME-DOMAIN TO THE FREQUENCY DOMAIN AS TYPICALLY THE CASE WITH AN N-pt. DFT (via an FFT)

20

- Note: each sinewave $s_k[n] = e^{j 2\pi \frac{k}{N} n}$ is periodic with period N

- For illustrative purposes, consider:



- For each sinewave, grab last L pts. and place them at the beginning as well to create sinewave of length N+L

- Due to superposition, one can first form sum of sinewaves of length N

$$x[n] = \sum_{k=0}^{N-1} b_k e^{j 2\pi \frac{k}{N} n} \{u[n] - u[n-N]\}$$

and then copy the last L values of x(n) and place them at the beginning to form:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} b_k e^{j 2\pi \frac{k}{N} n} \{u[n+L] - u[n-N]\}$$

- The first L values of $\tilde{x}[n]$ are called the cyclic prefix

- Why do it this way? Because the sum of sinewaves of length N

$$x[n] = \sum_{k=0}^{N-1} b_k \, e^{j 2\pi \frac{k}{N} n} \qquad n = 0, 1, \ldots, N-1$$

can be efficiently computed via an N-pt inverse FFT of $b_k = b[k]$ if N is a power of two

- Similar to before, don't think of the inverse FFT as transforming back from the frequency domain to the time domain as typically the case
- Here, inverse FFT efficiently generates the sum of sinewaves (collectively)

- Finally, we keep transmitting one OFDM block after another in succession

$\Rightarrow$ we can let the blocks of length $N+L$ bump up right up against each other DESPITE the fact the convolution with $h(n)$ will cause one block to overlap by $L-1$ pts. with the next block BECAUSE the part where they overlap is the part we throw away!

- the "transient" partial overlap pts. at the end of one block spill into the "transient" partial overlap pts. at the beginning of the next block $\Rightarrow$ discard!