Time-Domain Aliasing for Causal Single-Pole vs. Multi-Pole Signals

- Signal have poles is generally infinite in length
  - although practically decays to zero if poles are inside unit circle

- If we sample the DTFT at N equi-spaced points over $0 < \omega < 2\pi$, the N pt. inverse DFT yields time-domain aliasing:

$$x_p[n] = \left\{ \sum_{\ell=-\infty}^{\infty} x[n+\ell N] \right\} \left\{ u[n] - u[n-N] \right\}$$

- note: if $x[n]$ is causal, $x[n-N]$ starts at $n = N \Rightarrow$ does not contribute in $n = 0, 1, \ldots, N-1$

- If the signal is causal, only shifts to the left by ℓN contribute over $n = 0, 1, \ldots, N-1$
- So, for causal signal:

$$x_p[n] = \sum_{\ell=0}^{\infty} x[n+\ell N] \{u[n] - u[n-N]\}$$

- Now, consider single-pole signal:

$$x[n] = a^n u[n] \xleftrightarrow{\text{DTFT}} X(\omega) = \frac{1}{1 - a e^{-j\omega}}$$

$$\sum_{\ell=0}^{\infty} a^{n+\ell N} u[n+\ell N] \cdot \{u[n] - u[n-N]\} \xleftrightarrow[N]{\text{DFT}} X_N(k) = \frac{1}{1 - a e^{j 2\pi \frac{k}{N}}}$$

$k = 0, 1, \ldots, N-1$

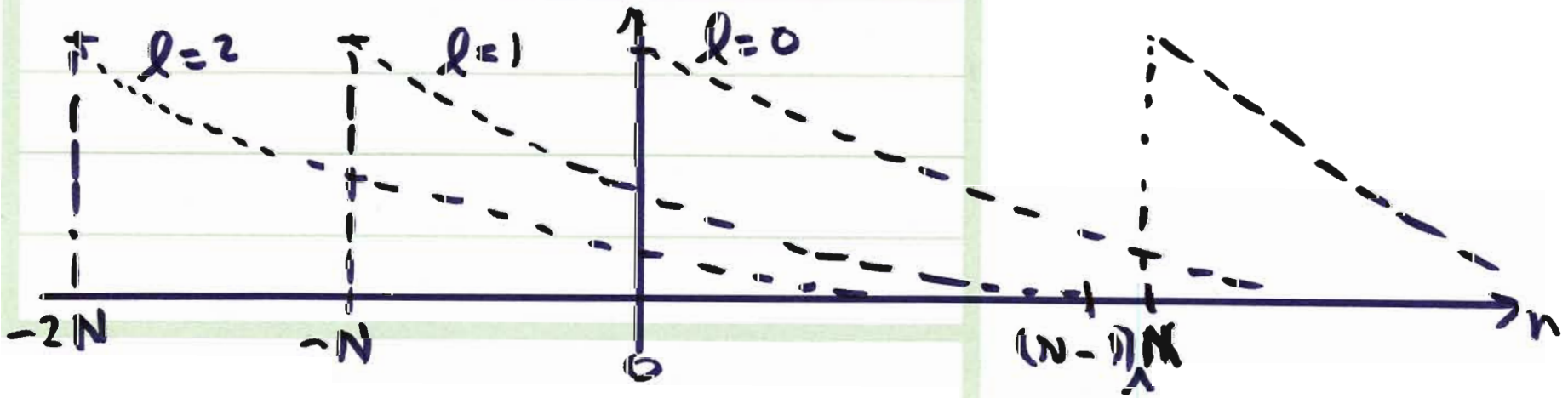
- Consider: $\sum_{\ell=0}^{\infty} a^{n+\ell N} = \left\{ \sum_{\ell=0}^{\infty} (a^N)^{\ell} \right\} a^n$

$$= a^n + a^N a^n + a^{2N} a^n + \cdots$$

$$= \left(\frac{1}{1 - a^N} \right) a^n$$

$\Rightarrow$ shape is preserved $\Rightarrow$

- time-domain aliasing only manifests itself as a multiplicative scalar



- But consider a two-pole signal

$$x[n] = c_1 a^n u[n] + c_2 b^n u[n]$$

- Time-domain aliasing:

$$x_p[n] = \sum_{\ell=0}^{\infty} x[n+\ell N] \{u[n] - u[n-N]\}$$

$\ell = 0$: $c_1 a^n + c_2 b^n$

$+ \ \ell = 1$: $(c_1 a^N) a^n + (c_2 b^N) b^n$

$+ \ \ell = 2$: $(c_1 a^{2N}) a^n + (c_2 b^{2N}) b^n$

$\vdots$

$$= c_1 \frac{1}{1-a^N} a^n + c_2 \frac{1}{1-b^N} b^n$$

the relative weightings change with each shift

$\Rightarrow$

shape is <u>not</u> preserved

- In old exams:

$$x[n] = \cos(\omega_0 n) = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$\cdot\, u[n] = \frac{1}{2} \left(e^{j\omega_0}\right)^n + \frac{1}{2} \left(e^{-j\omega_0}\right)^n$$

⇒ two-pole signal ⇒ time-domain aliasing alters shape

$$x[n] = a^{|n|} = a^n u[n] + a^{-n} u[-n-1]$$

$$= a^n u[n] + \left(\frac{1}{a}\right)^n u[-n-1]$$

⇒ two-pole signal ⇒ time-domain aliasing alters shape

⇒ basis for Matlab assignment 3