Noble's Identity II

- The two DT systems below have identical I/O relationships

Time Domain:

(A) $x[n] \rightarrow \boxed{M\downarrow} \rightarrow \boxed{h[n]} \rightarrow y[n]$

(B) $x[n] \rightarrow \boxed{f[n]} \rightarrow \boxed{M\downarrow} \rightarrow y[n]$

Identical I/O relationships

If: $f[n] = \sum_{k=-\infty}^{\infty} h[k]\,\delta[n-kM] \xleftrightarrow{\text{DTFT}} F(\omega) = H(M\omega)$

Frequency Domain Proof:

System A: $Y(\omega) = H(\omega)\frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega - k2\pi}{M}\right)$

System B: $Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} F\left(\frac{\omega - k2\pi}{M}\right) X\left(\frac{\omega - k2\pi}{M}\right)$

• If $F(\omega) = H(M\omega)$, then

$$F\left(\frac{\omega - k2\pi}{M}\right) = H\left(M\left(\frac{\omega - k2\pi}{M}\right)\right)$$

$$= H(\omega - k2\pi) = H(\omega)$$

• since DTFT is periodic with period 2π

• so with $F(\omega) = H(M\omega)$, system B's I/O is also:

$$Y(\omega) = H(\omega)\,\frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega - k2\pi}{M}\right)$$

• End of Proof

Application/Example/Typical Use

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow (M_1\downarrow) \longrightarrow \boxed{h_2[n]} \longrightarrow (M_2\downarrow) \longrightarrow y(n)$$

• Use Noble II to move 1st decimator to other side of 2nd filter

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{f_2[n]} \longrightarrow \boxed{M_1 \downarrow} \longrightarrow \boxed{M_2 \downarrow} \longrightarrow y[n]$$

- $$f_2[n] = \sum_{k=-\infty}^{\infty} h_2[k]\,\delta[n - kM_1]$$

- reduces to:

product of decimation factors

$$x[n] \longrightarrow \boxed{h_{eq}[n]} \longrightarrow \boxed{M_1 M_2 \downarrow} \longrightarrow y[n]$$

- where:

$$h_{eq}[n] = h_1[n] * f_2[n] \overset{DTFT}{\longleftrightarrow} H_{eq}(\omega) = H_1(\omega)\,H_2(M_1\omega)$$

$$h_2[n] \longrightarrow \boxed{M_1 \uparrow} \longrightarrow f_2[n]$$

- insert $M_1 - 1$ zeroes between each pair of successive values of $h_2[n]$ to form $f_2[n]$

Noble I Identity

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow (M\uparrow) \longrightarrow y[n] \quad \text{A}$$

$$x[n] \longrightarrow (M\uparrow) \longrightarrow \boxed{f[n]} \longrightarrow y[n] \quad \text{B}$$

A and B: Identical I/O Relationships

where: $f[n] = \sum_{k=-\infty}^{\infty} h[k]\,\delta[n-kM]$

$$h[n] \longrightarrow (M\uparrow) \longrightarrow f[n] \xleftrightarrow{\text{DTFT}} F(\omega) = H(M\omega)$$

Frequency Domain Proof

System A I/O $\Rightarrow$ $Y(\omega) = H(M\omega)X(M\omega)$

System B I/O $\Rightarrow$ $Y(\omega) = X(M\omega)F(\omega)$

If $F(\omega) = H(M\omega)$ $\Rightarrow$ systems have same I/O