• On Symmetric and Anti-Symmetric Linear Phase FIR Filters

• Recall symmetry properties of the DTFT

• $x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$

real-valued and even $x[-n] = x[n]$ — real-valued and even

• Consider $h_0[n]$ only nonzero for $|n| \leq L$ so that it is of "length" $M = 2L+1$ (FIR)

• $h[n] = h_0[n-L] = h_0[n - \frac{M-1}{2}]$ is causal

and $H(\omega) = \underbrace{H_0(\omega)}_{\text{real-valued}} e^{-j\frac{M-1}{2}\omega}$ ← linear phase

• $M = 2L+1$ is odd, so $L = \frac{M-1}{2}$ is integer

- Alternatively, note if $\boxed{\begin{array}{c} h_o[-n] = h_o[n] \\ n \in \{-L, \ldots, -1, 0, 1, \ldots, L\} \end{array}}$ ②

$$H_o(\omega) = \sum_{n=-L}^{L} h_o[n] e^{-j\omega n}$$

$$= h_o[0] + (h_o[1]\left(e^{-j\omega} + e^{-j\omega}\right)$$

$$+ h_o[2]\left(e^{j\omega 2} + e^{-j2\omega}\right)$$

$$\vdots$$

$$+ h_o[L]\left(e^{j\omega L} + e^{-j\omega L}\right)$$

$$\left.= h_o[0] + 2\sum_{k=1}^{L} h_0[k]\cos(k\omega)\right\} \text{real-valued}$$

Then

$$H(\omega) = H_o(\omega) e^{-jL\omega} = H_o(\omega) e^{-j\frac{(M-1)}{2}\omega}$$

for $h[n] = h_o[n-L] = h_o[n - \frac{M-1}{2}]$

- Similarly, for anti-symmetric case:

$x[n]$ real-valued and anti-symmetric $x[-n] = -x[n]$ $\overset{\text{DTFT}}{\longleftrightarrow}$ $X(\omega)$ purely imaginary (no real part)

- Consider $h_o[n]$ anti-symmetric such that $h_o[-n] = -h_o[n]$ over $|n| < L$ (FIR length $M = 2L+1$)

$\Rightarrow$ implies $h_o[0] = 0$

$$H_o(\omega) = \sum_{n=-L}^{L} h_o[n]\, e^{-j\omega n}$$

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$$H_o(w) = h_o[1]\left(e^{-jw} - e^{jw}\right)$$
$$+ h_o[2]\left(e^{-j2w} - e^{j2w}\right)$$
$$\vdots$$
$$+ h_o[L]\left(e^{-jLw} - e^{jLw}\right)$$

$$H_o(w) = -2j\sum_{k=1}^{L} h_o[k]\sin(kw) \quad \Big\}\ \text{purely imaginary}$$

Then:

$$h[n] = h_o\left[n - \tfrac{M-1}{2}\right] \xleftrightarrow{\text{DTFT}} H_o(w)\, e^{-j\frac{M-1}{2}w}$$
$$= j\, G_o(w)\, e^{-j\frac{M-1}{2}w}$$

where:

$$G_o(w) = -2\sum_{k=1}^{L} h_o[k]\sin(kw) \quad \Big\}\ \text{purely real-valued}$$

- Consider symmetric case with M even

$h_0[n]$ is nonzero for $\underbrace{-(L-1), \ldots, -1, 0, 1, \ldots, L}_{\text{length } L-1+L+1}$

$M = 2L$

$h_0[-n] = h_0[n+1]$

e.g.: $h_0[0] = h_0[1]$

$h_0[-1] = h_0[2]$

$\vdots$

$h_0[-(L-1)] = h_0[L]$

- DTFT: $H_0(\omega) = h_0[0] + h_0[1]\, e^{-j\omega}$

$+ h_0[-1]\, e^{j\omega} + h_0[2]\, e^{-j2\omega}$

$+ h_0[-2]\, e^{j2\omega} + h_0[3]\, e^{-j3\omega}$

$\cdots + h_0[-(L-1)]\, e^{j(L-1)\omega} + h_0[L]\, e^{-jL\omega}$

$$H_0(\omega) = h_0[1]e^{-j\frac{\omega}{2}}\left\{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right\}$$

$$+ h_0[2]e^{-j\frac{\omega}{2}}\left\{e^{j\frac{3\omega}{2}} + e^{-j\frac{3}{2}\omega}\right\}$$

$$+ h_0[3]e^{-j\frac{\omega}{2}}\left\{e^{j\frac{5}{2}\omega} + e^{-j\frac{5}{2}\omega}\right\}$$

$$\vdots$$

$$+ h_0[L]e^{-j\frac{\omega}{2}}\left\{e^{j(L-\frac{1}{2})\omega} + e^{-j(L-\frac{1}{2})\omega}\right\}$$

$$H_0(\omega) = \underbrace{\left\{2\sum_{k=1}^{L} h_0[k]\cos\left(\left(k-\tfrac{1}{2}\right)\omega\right)\right\}}_{\text{purely real-valued}}\underbrace{e^{-j\frac{\omega}{2}}}_{\text{corresponds to half-sample delay}}$$

$$h[n] = h_0[n-(L-1)] \overset{\text{DTFT}}{\longleftrightarrow} H_0(\omega)e^{-j(L-1)\omega}$$

• Recall $M = 2L \Rightarrow L = \frac{M}{2}$ 7

$$H(\omega) = G_0(\omega)\, e^{-j\left(\frac{M}{2}-1\right)\omega}\, e^{-j\frac{\omega}{2}}$$

$$= G_0(\omega)\, e^{-j\frac{(M-1)}{2}\omega}$$

where:

$$G_0(\omega) = 2\sum_{k=1}^{L} h_0[k]\cos\left(\left(k-\tfrac{1}{2}\right)\omega\right)$$

$$= 2\sum_{k=1}^{L} h[k+(L-1)]\cos\left(\left(k-\tfrac{1}{2}\right)\omega\right)$$

$$= 2\sum_{k=1}^{L} h\left[k+\frac{(M-2)}{2}\right]\cos\left(\left(k-\tfrac{1}{2}\right)\omega\right)$$