Quick note on Linear MMSE:

Minimum Mean Square Estimation

- Consider estimating Y in terms of three othe random variables (r.v.'s) linearly as

$$\hat{Y} = a_1 X_1 + a_2 X_2 + a_3 X_3$$

$$\mathcal{E} = Y - \hat{Y} \Rightarrow \min_{a_1, a_2, a_3} E\{\mathcal{E}^2\} = E\left((Y-\hat{Y})^2\right)$$

$$\frac{\partial}{\partial a_1}\left\{E\{(Y-a_1X_1-a_2X_2-a_3X_3)^2\}\right\}$$

$$= E\left\{\frac{\partial}{\partial a_1}(Y-a_1X_1-a_2X_2-a_3X_3)^2\right\}$$

$$= E\Big\{2\underbrace{(Y-a_1X_1-a_2X_2-a_3X_3)}_{\mathcal{E}\ \text{(error)}}(-X_1)\Big\} = 0$$

Similarly, $\frac{\partial}{\partial a_2}$ and $\frac{\partial}{\partial a_3}$ yield: collectively ②

$$\left.\begin{aligned} E\{\varepsilon X_1\} &= 0 \\ E\{\varepsilon X_2\} &= 0 \\ E\{\varepsilon X_3\} &= 0 \end{aligned}\right\}$$

"error ε is orthogonal to data"

X_1, X_2, X_3 are the "data" used to estimate Y

Ultimately, the normal eqns:

$$\begin{bmatrix} E(X_1X_1) & E(X_1X_2) & E(X_1X_3) \\ E(X_2X_1) & E(X_2X_2) & E(X_2X_3) \\ E(X_3X_1) & E(X_3X_2) & E(X_3X_3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} E(YX_1) \\ E(YX_2) \\ E(YX_3) \end{bmatrix}$$

$$\underline{R}_{xx}\,\underline{a} = \underline{r}_{yx} \Rightarrow \text{Wiener-Hopf Equations}$$

- Wiener-Hopf Equations

$$\underbrace{\underline{R}_{xx}}_{\text{autocorrelation matrix of "data"}} \; \underline{a} = \underbrace{\underline{r}_{yx}}_{\text{cross-correlation vector between quantity to be estimated and "data"}}$$

- again: X_1, X_2, X_3 are the "data" used to estimate Y