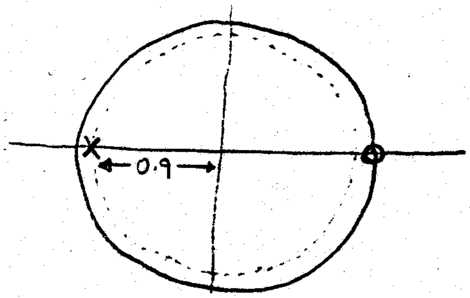
4.51

a)



High Pass $\Rightarrow$ Pole at $\omega = \pm\pi$ (at $r = 0.9$)

Constant Signals do not pass $\Rightarrow$ Zero at $\omega = 0$

$$H(z) = \frac{G(z-1)}{z+0.9} \qquad G\text{ - constant}$$

b) $$H(\omega) = \frac{G(e^{j\omega}-1)}{e^{j\omega}+0.9} = \frac{G e^{j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}{e^{j\omega}+0.9}$$

$$= \frac{2G\, e^{j\frac{\omega}{2}}\left(\sin\frac{\omega}{2}\right)\cdot j}{(\cos\omega + 0.9) + j\sin\omega}$$

$$|H(\omega)| = \frac{2|G|\cdot|\sin\frac{\omega}{2}|}{\sqrt{(\cos\omega+0.9)^2 + (\sin\omega)^2}}$$

$$\measuredangle H(\omega) = \frac{\omega}{2} - \tan^{-1}\frac{\sin\omega}{\cos\omega+0.9} + \frac{\pi}{2}$$

c) $|H(\pi)| = 1$

$$\frac{2G\sin\frac{\pi}{2}}{\sqrt{(\cos\pi+0.9)^2 + (\sin\pi)^2}} = 1$$

$$\frac{2G}{\sqrt{(-1+0.9)^2 + 0}} = 1$$

$$2G = 0.1$$

$$\boxed{G = 0.05} \qquad \Rightarrow H(\omega) = \frac{0.1\, e^{j\frac{\omega}{2}}\left(\sin\frac{\omega}{2}\right)\cdot j}{e^{j\omega}+0.9}$$

②

d) $$\frac{Y(z)}{X(z)} = \frac{0.05\,(z-1)}{z+0.9} = \frac{0.05 - 0.05z^{-1}}{1 + 0.9z^{-1}}$$

$$Y(z) + 0.9z^{-1}Y(z) = 0.05\,X(z) - 0.05z^{-1}X(z)$$

$$y(n) + 0.9\,y(n-1) = 0.05\,x(n) - 0.05\,x(n-1)$$

$$y(n) = -0.9\,y(n-1) + 0.05\,x(n) - 0.05\,x(n-1)$$

e) $$x(n) = 2\cos\left(\frac{\pi n}{6} + 45^\circ\right)$$

$$y(n) = 2\left|H\left(\frac{\pi}{6}\right)\right|\cos\left(\frac{\pi n}{6} + 45^\circ + \measuredangle H\left(\frac{\pi}{6}\right)\right)$$

$$= 2\,(0.0141)\cos\left(\frac{\pi n}{6} + 45^\circ + 89.19^\circ\right)$$

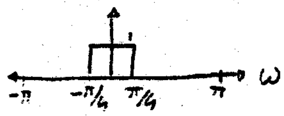
4.32

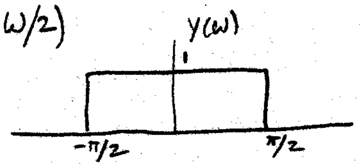
a) $y(n) = x(2n)$

Linear, Not Time Invariant

$Y(\omega) = X(\omega/2)$

Input =



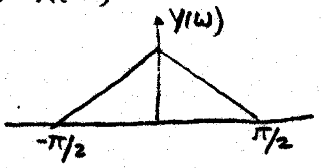


It has introduced new frequencies

b) $y(n) = x^2(n)$

Non Linear, Time Invariant

$Y(\omega) = X(\omega) * X(\omega)$

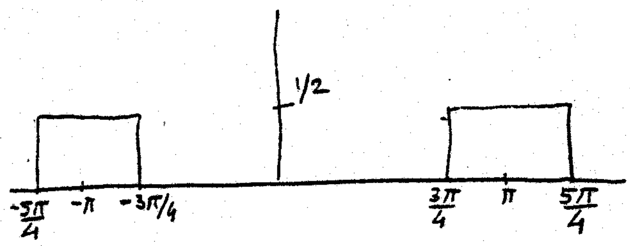


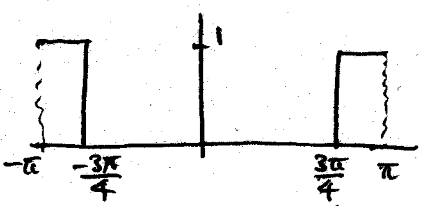
It has introduced new frequencies

c) $y(n) = \cos(\pi n)\,x(n)$

Linear, Not Time Invariant

$Y(\omega) = \frac{1}{2}X(\omega+\pi) + \frac{1}{2}X(\omega-\pi)$

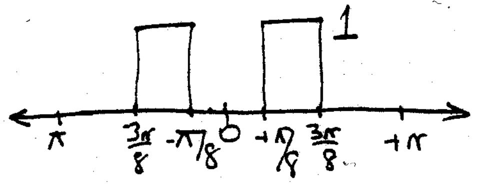




It has introduced new frequencies

4.47

Frequency response of an ideal bandpass filter is: ③



a) Determine the impulse response.

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{3\pi}{8}}^{-\pi/8} e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\pi/8}^{\frac{3\pi}{8}} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{jn} e^{-j\frac{\pi}{8}n} - \frac{1}{jn} e^{-j\frac{3\pi}{8}n}\right] + \frac{1}{2\pi}\left[\frac{1}{jn} e^{j\frac{3\pi}{8}n} - \frac{1}{jn} e^{j\frac{\pi}{8}n}\right]$$

$$= \frac{1}{\pi n}\sin\frac{3\pi}{8}n - \frac{1}{\pi n}\sin\frac{\pi}{8}n \qquad \mathrm{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$= \frac{3}{8}\,\mathrm{sinc}\left(\frac{3\pi}{8}n\right) - \frac{1}{8}\,\mathrm{sinc}\left(\frac{\pi}{8}n\right)$$

(Or using $\sin A - \sin(B) = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

$$= \frac{2}{\pi n}\sin\left(\frac{\pi}{8}n\right)\cos\left(\frac{\pi}{4}n\right) = \frac{2}{8}\,\mathrm{sinc}\left(\frac{1}{8}n\right)\cos\left(\frac{\pi}{4}n\right))$$

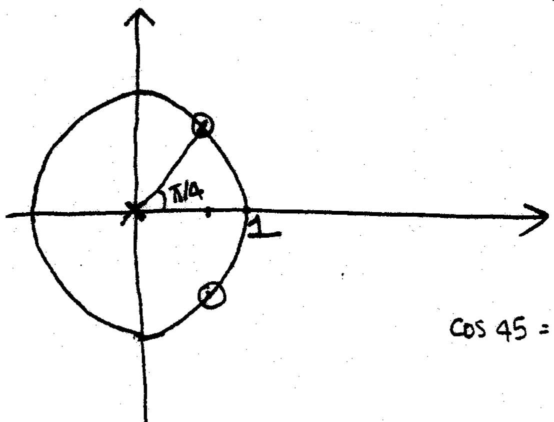
b) $h(n) = h_{LP}(n)\cos\frac{\pi}{4}n$

where $h_{LP}(n) = \frac{1}{4}\,\mathrm{Sinc}\left(\frac{1}{8}n\right)$

4.49

Find $|X(\omega)|$ corresponding to following

a)



find $|X(\omega)|$ at a few points

$$X(0) = \frac{|d||d|}{|1||1|} = (0.765)^2 = 0.586$$

$$\cos 45 = \sin 45 = \frac{1}{\sqrt{2}} = a$$

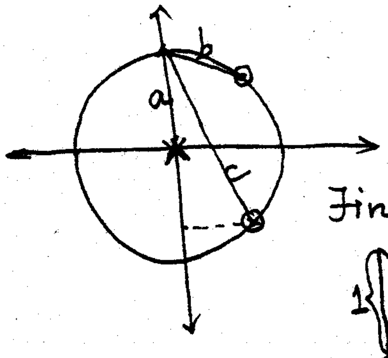
$$\therefore d = \sqrt{a^2 + (1-a)^2}$$

$$= \sqrt{\frac{1}{2} + \left(1 - \frac{1}{\sqrt{2}}\right)^2}$$

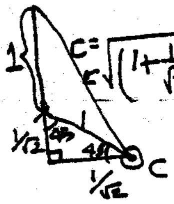
$\omega = \frac{\pi}{4}, -\frac{\pi}{4}. \quad |X(\omega)| = 0$ (4)

from before (same distance)

$\omega = \frac{\pi}{2}. \quad |X(\omega)| = \frac{|b|\,|c|}{|a|\,|a|} = \frac{0.765 \times \sqrt{2+\frac{2}{\sqrt{2}}}}{1.1} = 1.413$

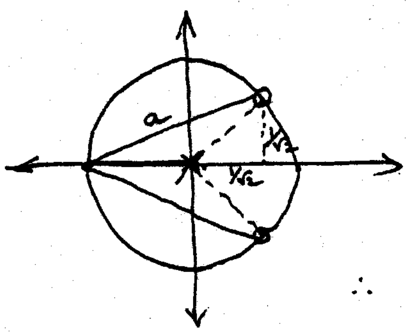


Find c :



$c = \sqrt{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1+\frac{2}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2}} = \sqrt{2+\frac{2}{\sqrt{2}}} = 1.848$

$\omega = \pi$

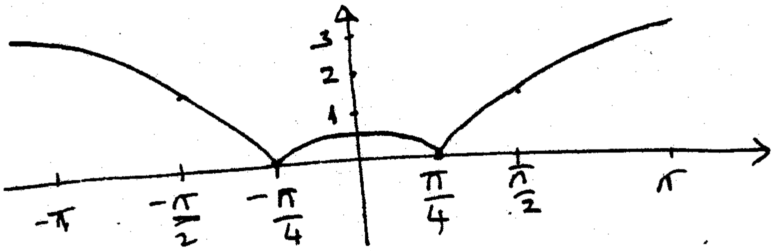


$X(\omega) = \frac{|a|^2}{1.1}$ where $a = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2}$

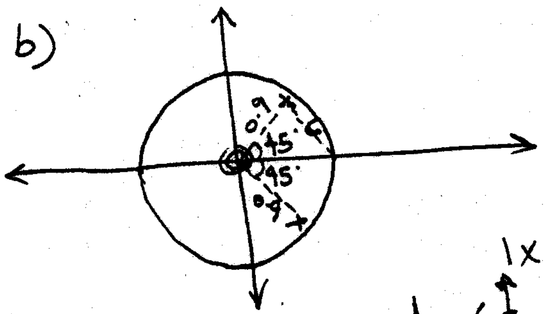
$= 1.848$

$X(\omega) = 3.414$

∴ Plot looks like this:

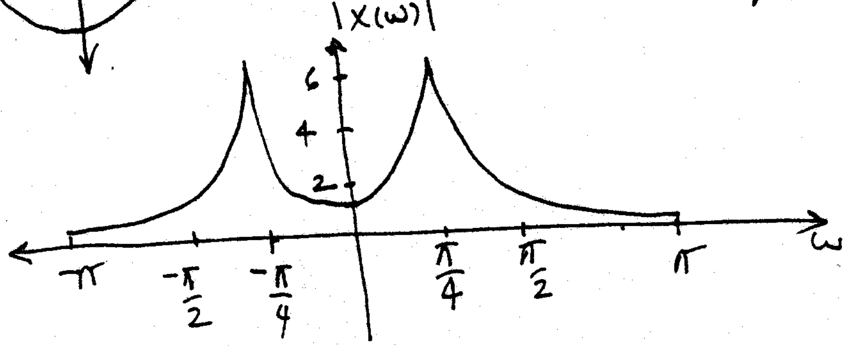


b)



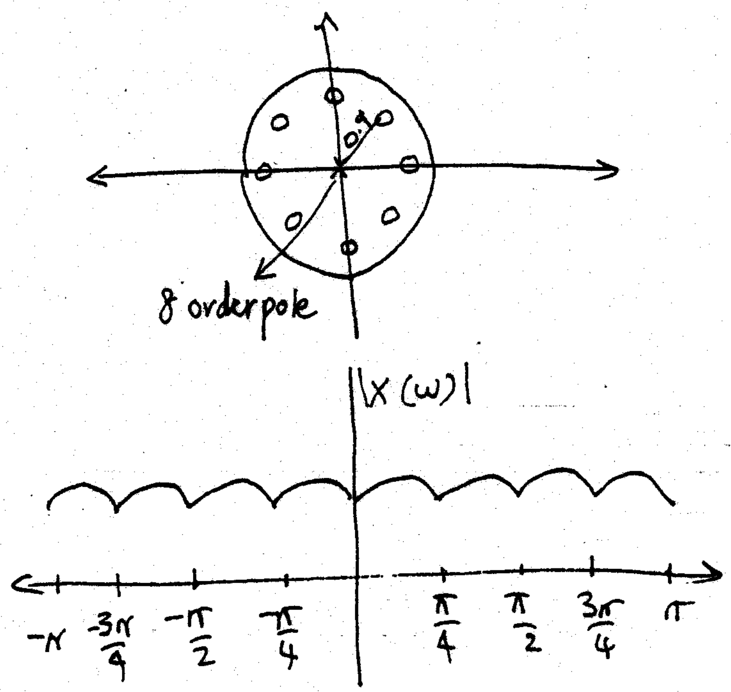
$|X(\omega)|$ at $\omega = 0 = \frac{1.1}{c^2}$

compute at few values like befores and obtain:

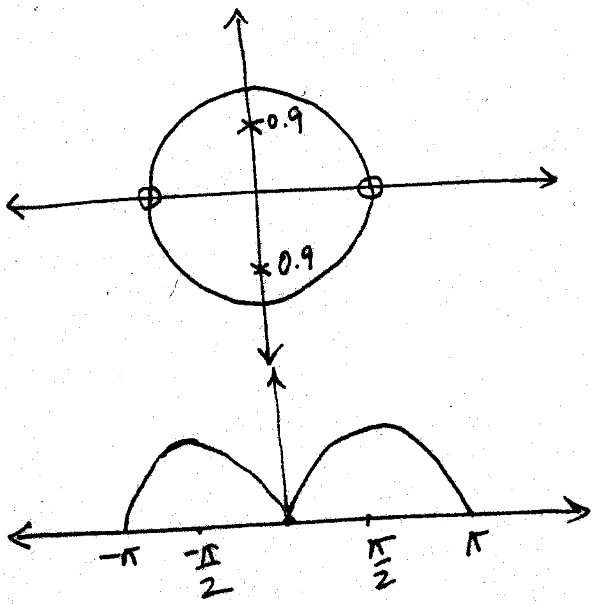


5

c)



d)



4.50 Design an FIR filter that completely blocks $\omega_0 = \pi/4$. compute output if input is $x(n) = \sin\left(\frac{\pi}{4}\right)n\, u(n)$. for $n = 0, 1, 2, 3, 4$.

$$H(z) = \left(1 - e^{j\frac{\pi}{4}} z^{-1}\right)\left(1 - e^{-j\frac{\pi}{4}} z^{-1}\right)$$

$$= 1 - e^{-j\frac{\pi}{4}} z^{-1} - e^{j\frac{\pi}{4}} z^{-1} + z^{-2}$$

$$= 1 - 2\cos\left(\frac{\pi}{4}\right) z^{-1} + z^{-2}$$

$$y(n) = x(n) - 2\left(\cos\frac{\pi}{4}\right)x(n-1) + x(n-2)$$

$x(0) = 0$, $x(1) = 0.707$ $x(2) = 1$ $x(3) = 0.707$ $x(4) = 0$

$y(0) = x(0) - 2 * 0.707 \times 0 + 0$

$= 0.$

$y(1) = x(1) - 2 \times 0.707 \times x(0) + 0$

$= 0.707$

$y(2) = x(2) - 2 \times 0.707 \times x(1) + x(0)$

$= 1 - 2 \times 0.707 \times 0.707 + 0$

$= 1 - \frac{2}{2} = 0.$

$y(3) = x(3) - 2 \times 0.707 . x(2) + x(1)$

$= 0.707 - 2 \times 0.707 \times 1 + 0.707$

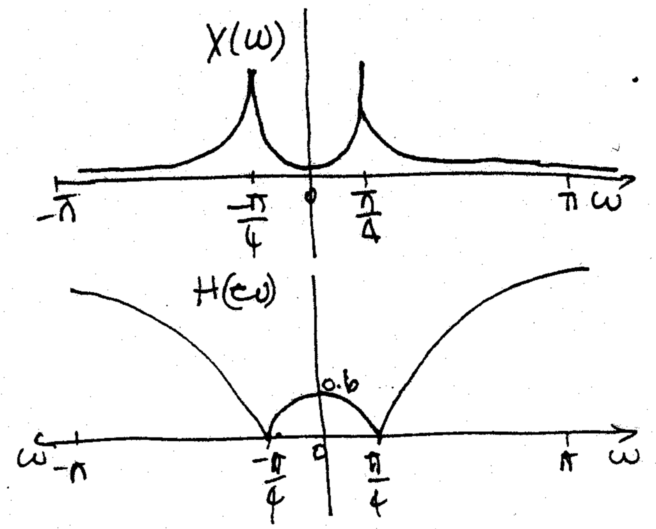
$= 0$

$y(4) = x(4) - 2 \times 0.707\, x(3) + x(2)$

$= 0 - 2 \times 0.707 \times 0.707 + 1$

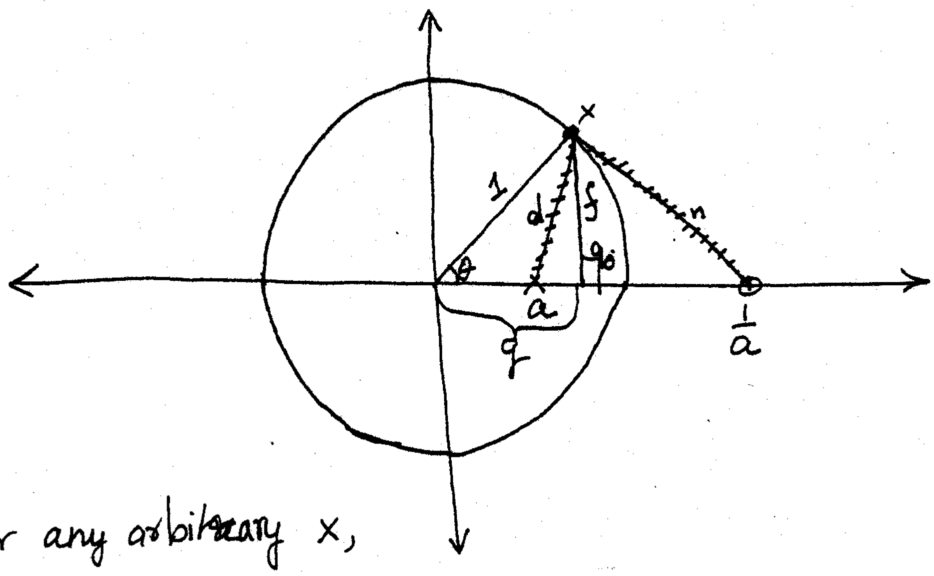
$= 0$

No it has a nonzero sample $y(1)$.
This is because the input is a onesided sequence.
If it was a pure sinusoid we would have $x(-1) = -0.707$ $x(-2) = -1 \ldots$
Then we would have had $y(n) = 0\ \forall n$.
The reason for the above phenomenon is that $X(\omega)$ is not zero for all ω ~~other than~~ $\frac{\pi}{4}$.



7

4.76. Show that the system specified by the
(a) following pole zero plot is all pass.
ie $H(\omega)$ is constant.



For any arbitrary x,

$$|H(\omega)| = \frac{|n|}{|d|}.$$

Construction: drop a perpendicular from x to horizontal axis. Call it f.
Line joining x and pole is d and line joining x and zero is n.

Note $\cos\theta = g$ $\sin\theta = f$.

$$|d| = \sqrt{f^2 + (g-a)^2} = \sqrt{\sin^2\theta + (\cos\theta - a)^2} = \sqrt{\sin^2\theta + \cos^2\theta - 2\cos\theta\, a + a^2}$$
$$= \sqrt{a^2 - 2\cos\theta\, a + 1}$$

$$|n| = \sqrt{f^2 + \left(\tfrac{1}{a} - g\right)^2}$$
$$= \sqrt{\sin^2\theta + \left(\tfrac{1}{a^2} - \tfrac{2\cos\theta}{a} + \cos^2\theta\right)}.$$
$$= \sqrt{\sin^2\theta + \cos^2\theta + \tfrac{1}{a^2} - \tfrac{2\cos\theta}{a}}$$
$$= \sqrt{1 + \tfrac{1}{a^2} - \tfrac{2\cos\theta}{a}}$$

$$\therefore \frac{|n|}{|d|} = \frac{\sqrt{a^2 + 1 - 2a\cos\theta}}{\sqrt{a^2(a^2 + 1 - 2a\cos\theta)}} = \frac{1}{a} \text{ (independent of } \theta\text{)}.$$

$\Rightarrow |H(\omega)|$ is independent of ω, therefore is a constant.

4.93

$$y(n) = \frac{1}{2} y(n-1) + x(n)$$

$$Y(z) = \frac{X(z)}{1 - \frac{1}{2} z^{-1}} \qquad h(n) = \left(\frac{1}{2}\right)^n u(n)$$

a) $\displaystyle r_{xx}(l) = \frac{1}{1 - \left(\frac{1}{4}\right)^2} \left(\frac{1}{4}^{|l|}\right)$ (Derive result 2.6.20)

b) $\displaystyle r_{hh}(l) = \frac{1}{1 - \left(\frac{1}{2}\right)^2} \left(\frac{1}{2}^{|l|}\right)$

c) $r_{xy}(l) = x(l) * y(-l)$

$$R_{xy}(z) = X(z)\, Y(z^{-1})$$

$$= X(z)\, X(z^{-1}) \frac{1}{1 - \frac{1}{2} z}$$

$$= \frac{1}{1 - \frac{1}{4} z^{-1}} \cdot \frac{1}{1 - \frac{1}{4} z} \cdot \frac{1}{1 - \frac{1}{2} z} = \frac{8z}{(z - \frac{1}{4})(z-4)(z-2)}$$

$$\frac{R_{xy}(z)}{z} = \frac{8}{(z - \frac{1}{4})(z-4)(z-2)} = \frac{A_1}{z - \frac{1}{4}} + \frac{A_2}{z-4} + \frac{A_3}{z-2}$$

$$A_1 = \frac{8}{(\frac{1}{4} - 4)(\frac{1}{4} - 2)} = \frac{8}{(-\frac{15}{4})(-\frac{7}{4})} = \frac{128}{105} \qquad A_2 = \frac{16}{15} \qquad A_3 = -\frac{16}{7}$$

$$R_{xy}(z) = \frac{128}{105} \left(\frac{1}{1 - \frac{1}{4} z^{-1}}\right) + \frac{16}{15} \left(\frac{1}{1 - 4z^{-1}}\right) - \frac{16}{7} \left(\frac{1}{1 - 2z^{-1}}\right)$$

$$r_{xy}(l) = \frac{128}{105} \left(\frac{1}{4}\right)^l u(l) - \frac{16}{15} 4^l u(-l-1) + \frac{16}{7} 2^l u(-l-1)$$

ROC
$(|z| > \frac{1}{4}) \cap |z| < 4 \cap (|z| < 2)$
$= \frac{1}{4} \le |z| < 2$

d) $r_{yy}(l) = y(l) * y(-l)$

$$R_{yy}(z) = Y(z)\, Y(z^{-1})$$

$$= X(z)\, X(z^{-1}) \frac{1}{1 - \frac{1}{2} z^{-1}} \frac{1}{1 - \frac{1}{2} z}$$

$$= \frac{1}{1 - \frac{1}{4} z} \cdot \frac{1}{1 - \frac{1}{4} z^{-1}} \cdot \frac{1}{1 - \frac{1}{2} z^{-1}} \cdot \frac{1}{1 - \frac{1}{2} z}$$

$$= \frac{8z^2}{(z-4)(z - \frac{1}{4})(z - \frac{1}{2})(z-2)}$$

$$\frac{R_{yy}(z)}{z} = \frac{8z}{(z-4)(z-\frac{1}{4})(z-2)(z-\frac{1}{2})}$$

$$= \frac{A_1}{z-4} + \frac{A_2}{z-\frac{1}{4}} + \frac{A_3}{z-2} + \frac{A_4}{z-\frac{1}{2}}$$

$$A_1 = \frac{8\times 4}{\frac{15}{4}\cdot 2\cdot\frac{7}{2}} = \frac{128}{105} \qquad A_2 = \frac{8\times\frac{1}{4}}{(-\frac{15}{4})(-\frac{7}{4})(-\frac{1}{4})} = -\frac{128}{105}$$

$$A_3 = \frac{8\times 2}{-2\cdot\frac{7}{4}\cdot\frac{3}{2}} = -\frac{64}{21} \qquad A_4 = \frac{8\times\frac{1}{2}}{-\frac{7}{2}\cdot\frac{1}{4}\cdot -\frac{3}{2}} = \frac{64}{21}$$

$$R_{yy}(z) = \frac{128}{105}\left(\frac{1}{1-4z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}\right) - \frac{64}{21}\left(\frac{1}{1-2z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}\right)$$

$$\left(\begin{aligned} \text{ROC} &= (|z|<4)\cap(|z|>\tfrac{1}{4})\cap(|z|<2)\cap(|z|>\tfrac{1}{2}) \\ &= \tfrac{1}{2}<|z|<2 \end{aligned}\right)$$

$$r_{yy}(l) = \frac{128}{105}\left(-4^l u(-l-1) - \left(\frac{1}{4}\right)^l u(l)\right) + \frac{64}{21}\left(2^l u(-l-1) + \left(\frac{1}{2}\right)^l u(l)\right)$$

4.100

a) Taking F.T.

$G(\omega) = H(\omega)X(\omega)$

$F(\omega) = H(\omega)G(-\omega) = H(\omega)H(-\omega)X(-\omega)$

$Y(\omega) = F(-\omega)$

$= H(-\omega)H(\omega)X(\omega)$

$= H^*(\omega)H(\omega)X(\omega)$ $\qquad H^*(\omega) = H(-\omega)$ $\because$ real sequence

$Y(\omega) = |H(\omega)|^2 X(\omega)$

Phase of this term is zero $\Rightarrow$ Zero phase filter

b) $G(\omega) = H(\omega)X(\omega)$

$F(\omega) = H(\omega)X(-\omega)$

$Y(\omega) = G(\omega) + F(-\omega)$

$= H(\omega)X(\omega) + H(-\omega)X(\omega) = [H(\omega) + H(-\omega)]X(\omega)$

$= [H(\omega) + H^*(\omega)]X(\omega)$

$= 2H_R(\omega)X(\omega)$

Phase of $2H_R(\omega)$ is zero ($\because Re(H(\omega))$) $\Longrightarrow$ Zero Phase Filter.