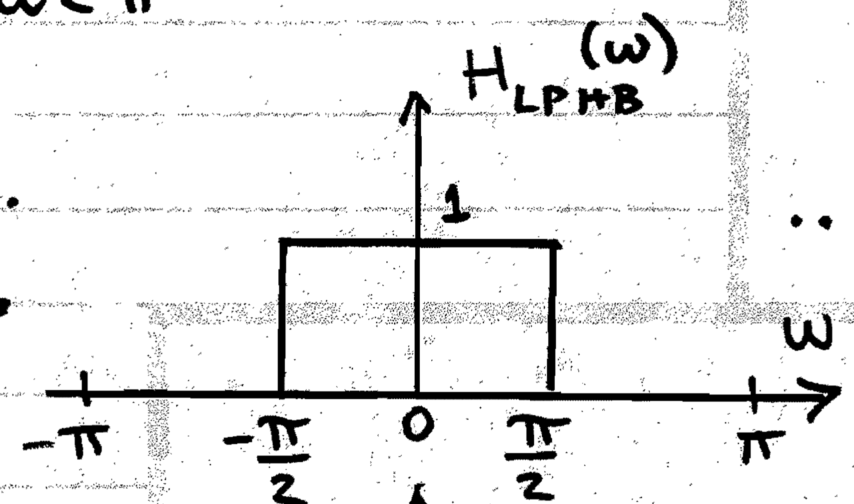
- Isolating positive frequency portion of spectrum in $0<\omega<\pi$ for bandwidth efficiency

- Use a complex-valued filter to pass only what's in $0<\omega<\pi$

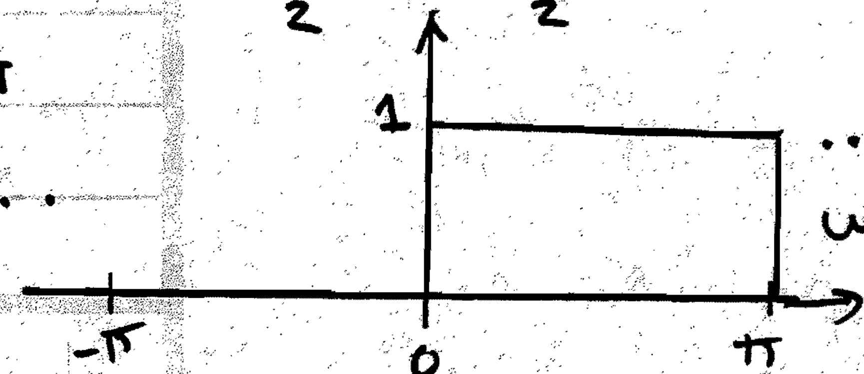
- For example:

$$h_{LPHB}(t) = \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} \xleftrightarrow{\text{DTFT}} \cdots$$

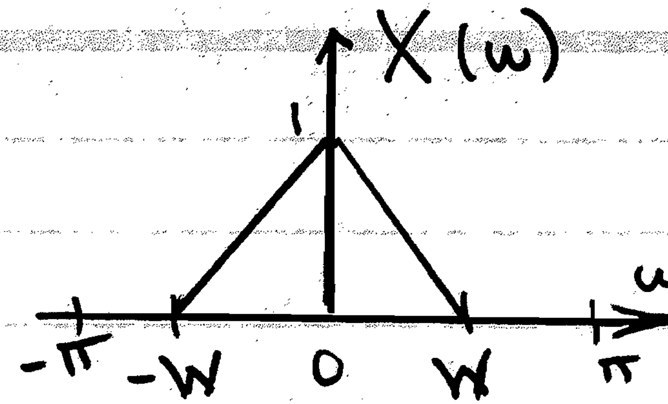


THEN:

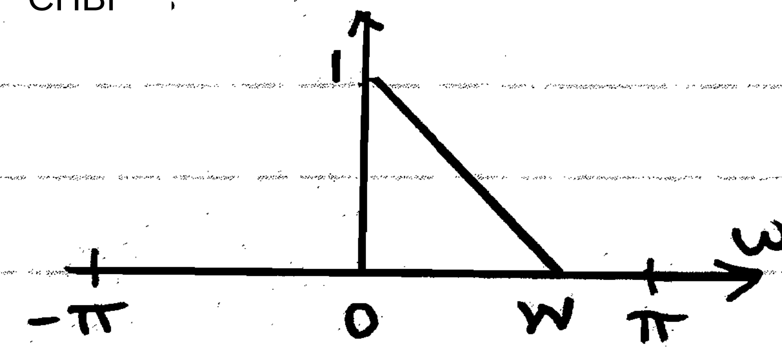
$$h_{\text{CHBF}}(t) = e^{j\frac{\pi}{2}n}\,\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} \xleftrightarrow{\text{DTFT}} \cdots$$



Thus: if $x[n] \xleftrightarrow{\text{DTFT}}$



then: $\tilde{x}[n] = x[n] * h_{\text{CHBF}}[n] \xleftrightarrow{\text{DTFT}}$



CHBF: complex halfband filter passing 0 to pi. Convention is to have the gain be 2 over 0 to pi so that the real part of the output is x[n] (as opposed to 0.5 x[n])We show this now, by first examining:

$$2e^{j\frac{\pi}{2}n}\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} = 2\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}\cos\left(\frac{\pi}{2}n\right) + j\,2\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}\sin\left(\frac{\pi}{2}n\right)$$

$$= \frac{\sin(\pi n)}{\pi n} + j\,2\,\frac{\sin^2\left(\frac{\pi}{2}n\right)}{\pi n}$$

$$= \delta[n] + j\,h_{HT}[n]$$

- Ideal Hilbert Transformer:

$$h_{HT}(t) = \frac{2\sin\left(\frac{\pi}{2}n\right)}{\pi n}\sin\left(\frac{\pi}{2}n\right)$$

- Hilbert Transform of $x[n]$:

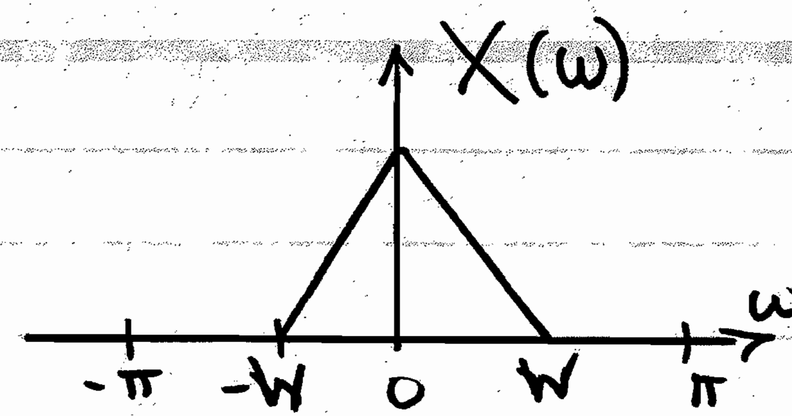
$$x[n] \longrightarrow \boxed{h_{HT}[n]} \longrightarrow \hat{x}[n] = x[n] * h_{HT}(t)$$

- Complex analytical signal:

$$\tilde{x}[n] = x[n] + j\,\hat{x}[n]$$

- In practice, only need complex filter to be unity (flat) over band signal occupies in $0 < \omega < \pi$ $\Rightarrow$ see Exam 2 Fall 2005

- for example: for



- $\tilde{x}[n] = 2x[n] * \left\{ e^{j\frac{W}{2}n} \frac{\sin\left(\frac{W}{2}n\right)}{\pi n} \right\}$

$$= x[n] * \frac{\sin(Wn)}{\pi n} + j \underbrace{x[n] * \frac{2\sin\left(\frac{W}{2}n\right)}{\pi n} \sin\left(\frac{W}{2}n\right)}_{\hat{x}[n]}$$

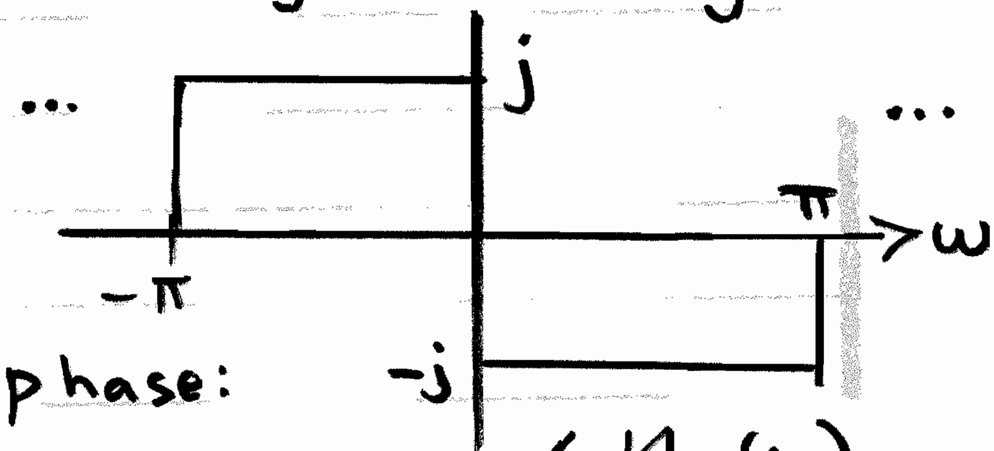
- See Exam 2, Fall 2005

• Let's examine frequency response of ideal Hilbert Transformer (HT)

• recall: $x[n]\sin(\omega_c n) \overset{DTFT}{\longleftrightarrow} \frac{1}{2j}X(\omega-\omega_c) - \frac{1}{2j}X(\omega+\omega_c)$

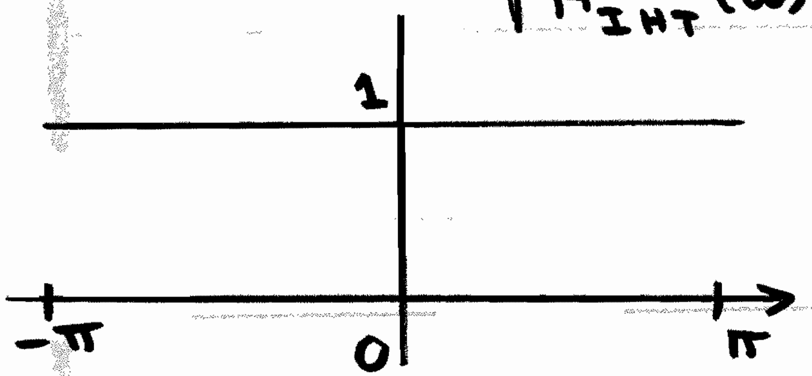
THUS:

$$2\,\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}\sin\left(\frac{\pi}{2}n\right) \overset{DTFT}{\longleftrightarrow}$$

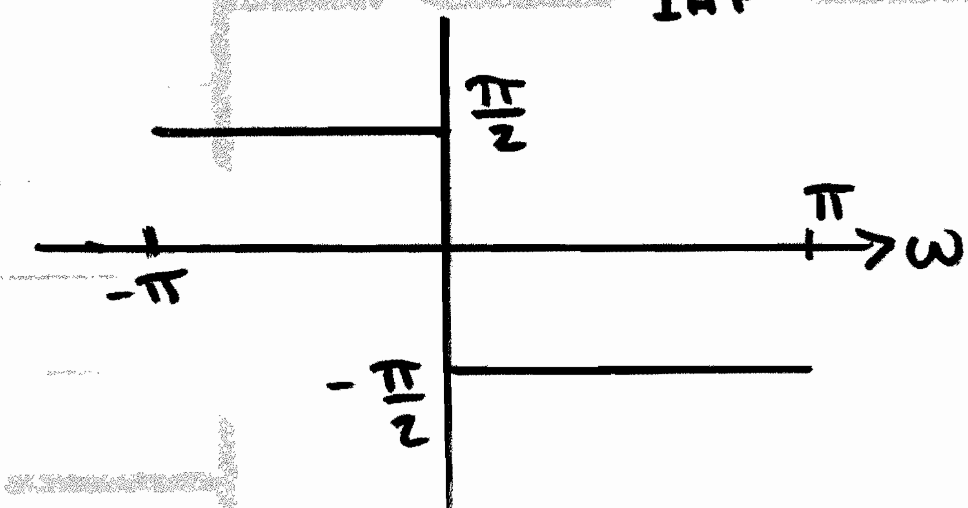


In terms of magnitude & phase:

$|H_{IHT}(\omega)|$



$\angle H_{IHT}(\omega)$



See Exam 2 Fall 2005

- Summarizing, for ideal case:

$$\tilde{x}[n] = x[n] * h_{CHBF}[n] \xleftrightarrow{DTFT} \tilde{X}(\omega) \begin{cases} = 0, & -\pi < \omega < 0 \\ = 2X(\omega), & 0 < \omega < \pi \end{cases}$$

$\boxed{\text{CHBF} \Rightarrow \text{complex halfband filter}}$

$$h_{CHBF}[n] = 2\,\frac{\sin(\frac{\pi}{2}n)}{\pi n}\, e^{j\frac{\pi}{2}n} = \delta[n] + j\underbrace{2\,\frac{\sin(\frac{\pi}{2}n)}{\pi n}\sin(\frac{\pi}{2}n)}_{h_{HT}[n]}$$

$$\tilde{x}[n] = x[n] * \delta[n] + j\, x[n] * h_{HT}[n]$$

$$= x[n] + j\,\hat{x}[n]$$

$$\mathrm{Re}\{\tilde{x}[n]\} = x[n]$$

Note:

$$\mathrm{Re}\{\tilde{x}[n]\} = \frac{1}{2}\{\tilde{x}[n] + \tilde{x}^*[n]\} \xleftrightarrow{DTFT} \frac{1}{2}\{\tilde{X}(\omega) + \underbrace{\tilde{X}^*(-\omega)}_{\text{restores negative frequency portion of spectrum}}\}$$

- Examine again frequency response for
  $h_{CHBF}[n] = \delta[n] + j\, h_{HT}[n]$

- where:

$$h_{HT}[n] = 2\,\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}\sin\left(\frac{\pi}{2}n\right) \overset{DTFT}{\longleftrightarrow} H_{HT}(\omega) = \begin{cases} +j, & -\pi < \omega < 0 \\ -j, & 0 < \omega < \pi \end{cases}$$

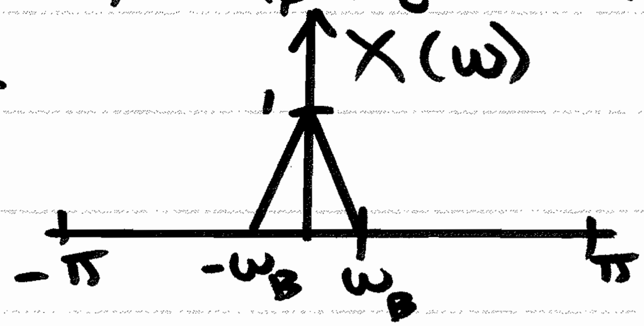
- THUS:

$$H_{CHBF}(\omega) = \begin{cases} 1 + j\,(j) = 1 - 1 = 0 & \text{for } -\pi < \omega < 0 \\ 1 + j\,(-j) = 1 + 1 = 2 & \text{for } 0 < \omega < \pi \end{cases}$$

- Thus, $h_{CHBF}[n]$ only passes positive frequency portion of overall spectrum (that was our starting point)
- <u>Keep in mind</u>: DTFT is always periodic with period 2π

- Suppose that due to either initial oversampling or digital upsampling that we have: (8)

$x[n] \overset{DTFT}{\longleftrightarrow}$



where:

$\omega_B << \pi$

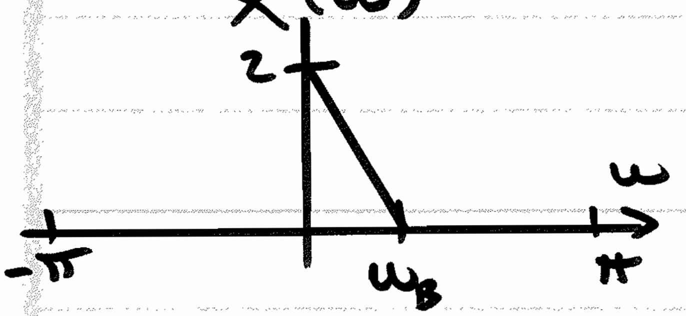
such that

$\omega_c + \omega_B < \pi$

Consider: $y[n] = Re\{e^{j\omega_c n}\tilde{x}[n]\}$

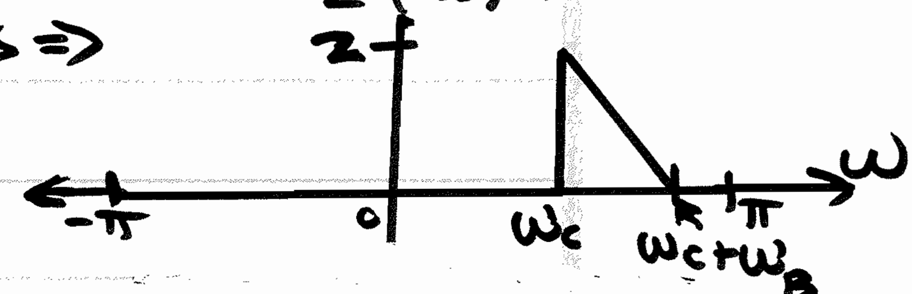
- First, $z[n] = e^{j\omega_c n}\tilde{x}[n]$
  where $\tilde{x}[n] \doteq x[n] + j\hat{x}[n]$ $\Big\}$ $\hat{x}[n] = x[n] * h_{HT}[n]$

$\tilde{X}(\omega)$



$\Rightarrow$ THUS $\Rightarrow$

$Z(\omega) = \tilde{X}(\omega - \omega_c)$



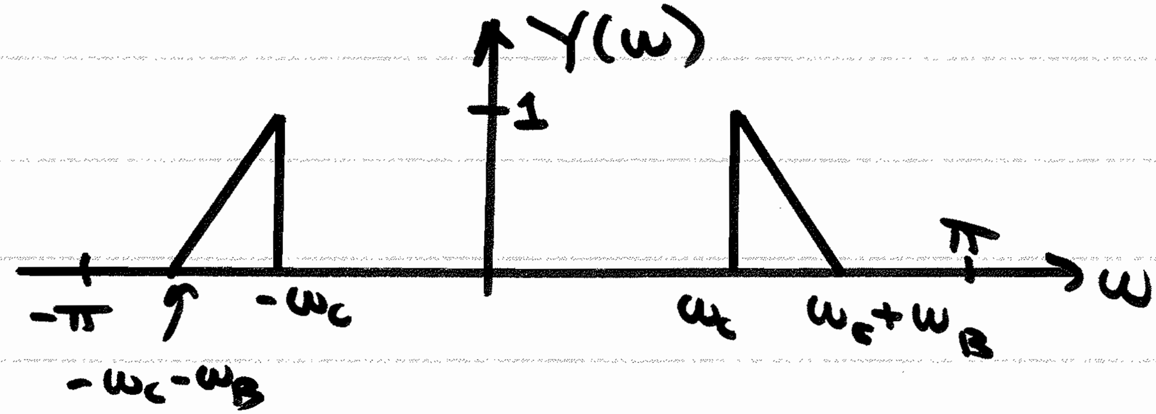
and $Re\{z[n]\} = \frac{1}{2}z[n] + \frac{1}{2}z^*[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{2}Z(\omega) + \frac{1}{2}Z^*(-\omega)$

- Thus, the DTFT of $y[n] = \mathrm{Re}\{e^{j\omega_c n}\tilde{x}[n]\}$

$$= \mathrm{Re}\{[\cos(\omega_c n) + j\sin(\omega_c n)][x[n] + j\hat{x}[n]]\}$$

$$= x[n]\cos(\omega_c n) - \hat{x}[n]\sin(\omega_c n) \xleftrightarrow{\text{DTFT}}$$

is:

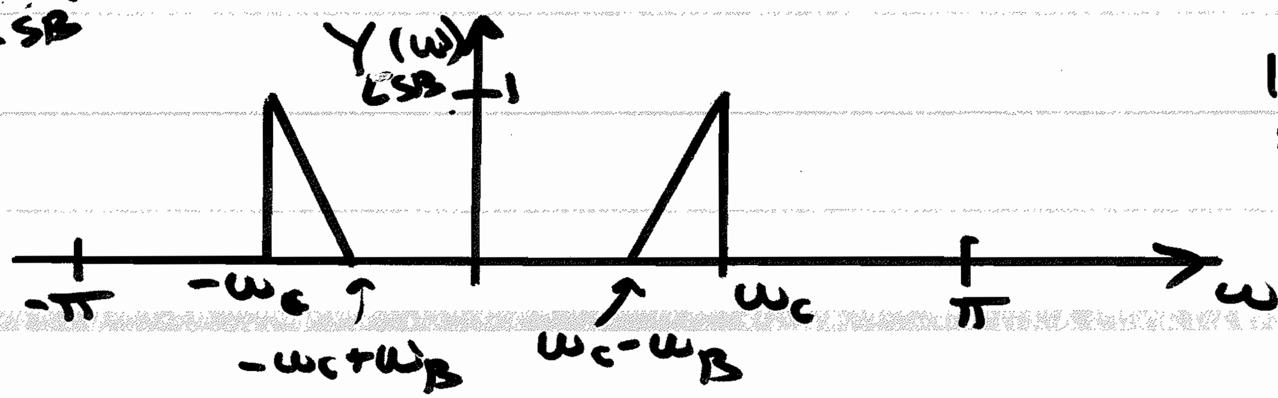


Single sideband modulation $\Rightarrow$ upper sideband

- Can easily show that DTFT of

$$y_{LSB}[n] = x[n]\cos(\omega_c n) + \hat{x}[n]\sin(\omega_c n)$$

is



lower sideband