1.5.4 ZT AND LINEAR, CONSTANT COEFFICIENT DIFFERENCE EQUATIONS

- From the convolution property, we obtain a characterization for all LTI systems

  - impulse response: $y(n) = h(n) * x(n)$
  - transfer function: $Y(z) = H(z)\, X(z)$

- An important class of LTI systems are those characterized by linear, constant coefficient difference equations

$$y(n) = \sum_{k=0}^{M} a_k \, x(n-k) - \sum_{\ell=1}^{N} b_\ell \, y(n-\ell)$$

  - nonrecursive
    $N = 0$
    always finite impulse response (FIR)

  - recursive
    $N > 0$
    usually infinite impulse response (IIR)

- Take ZT of both sides of equation

$$y(n) = \sum_{k=0}^{M} a_k \, x(n-k) - \sum_{\ell=1}^{N} b_\ell \, y(n-\ell)$$

$$Y(z) = \sum_{k=0}^{M} a_k \, z^{-k} \, X(z) - \sum_{\ell=1}^{N} b_\ell \, z^{-\ell} \, Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k \ z^{-k}}{1 + \sum_{\ell=1}^{N} b_\ell \ z^{-\ell}}$$

– $M \geq N$

multiply numerator and denominator by z^M

$$H(z) = \frac{\sum_{k=0}^{M} a_k \ z^{M-k}}{z^{M-N}\left[z^N + \sum_{\ell=1}^{N} b_\ell \ z^{N-\ell}\right]}$$

– $M < N$

multiply numerator and denominator by z^N

$$H(z) = \frac{z^{N-M} \sum_{k=0}^{M} a_k \, z^{M-k}}{z^N + \sum_{\ell=1}^{N} b_\ell \, z^{N-\ell}}$$

- By the fundamental theorem of algebra, the numerator and denominator polynomials may always be factored
  - $M \geq N$

$$H(z) = \frac{\prod_{k=1}^{M} (z - z_k)}{z^{M-N} \prod_{\ell=1}^{N} (z - p_\ell)}$$

– $M < N$

$$H(z) = \frac{z^{N-M} \prod_{k=1}^{M} (z - z_k)}{\prod_{\ell=1}^{N} (z - p_\ell)}$$

- Roots of the numerator and denominator polynomials
  - zeros $z_1, \ldots, z_M$
  - poles $p_1, \ldots, p_N$

- If $M \geq N$, have $M - N$ additional poles at $|z| = \infty$
- If $M < N$, have $N - M$ additional zeros at $z = 0$

- The poles and zeros play an important role in determining system behavior.

Example

$$y(n) = x(n) + x(n-1) - \frac{1}{2}\, y(n-2)$$

$$Y(z) = X(z) + z^{-1}\, X(z) - \frac{1}{2}\, z^{-2}\, Y(z)$$

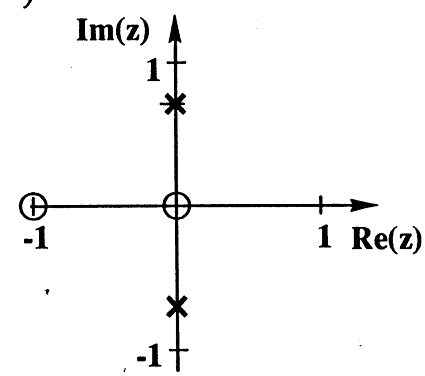
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 + \frac{1}{2}\, z^{-2}}$$

$$H(z) = \frac{z(z+1)}{z^2 + 1/2}$$

$$= \frac{z(z+1)}{(z - j/\sqrt{2})(z + j/\sqrt{2})}$$

zeros: $z_1 = 0$ $z_2 = -1$

poles: $p_1 = j/\sqrt{2}$ $p_2 = -j/\sqrt{2}$



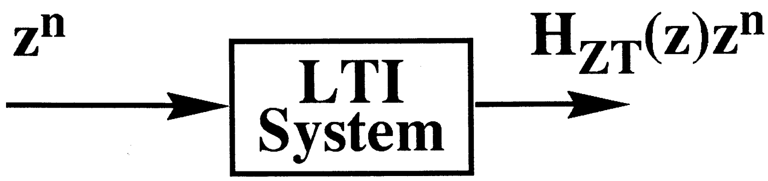
Effect of Poles and Zeros on Frequency Response

Frequency response $H(e^{j\omega})$

$$h(n) \overset{\text{DTFT}}{\longleftrightarrow} H_{DTFT}(e^{j\omega}) = H(e^{j\omega})$$

$$h(n) \overset{\text{ZT}}{\longleftrightarrow} H_{ZT}(z)$$

$$H_{DTFT}(e^{j\omega}) = H_{ZT}(e^{j\omega})$$



Let $z = e^{j\omega}$

$$\Rightarrow \; H(e^{j\omega}) = H_{ZT}(e^{j\omega})$$

Assume $M < N$

$$H(z) = \frac{z^{N-M} \prod_{k=1}^{M} (z - z_k)}{\prod_{\ell=1}^{N} (z - p_\ell)}$$

$$H(e^{j\omega}) = \frac{e^{j\omega(N-M)} \prod_{k=1}^{M} (e^{j\omega} - z_k)}{\prod_{\ell=1}^{N} (e^{j\omega} - p_\ell)}$$

$$|H(e^{j\omega})| = \frac{\prod_{k=1}^{M} |e^{j\omega} - z_k|}{\prod_{\ell=1}^{N} |e^{j\omega} - p_\ell|}$$

$$\angle H(e^{j\omega}) = \omega(N - M) + \sum_{k=1}^{M} \underline{/e^{j\omega} - z_k}$$

$$- \sum_{\ell=1}^{N} \underline{/e^{j\omega} - p_\ell}$$

Example

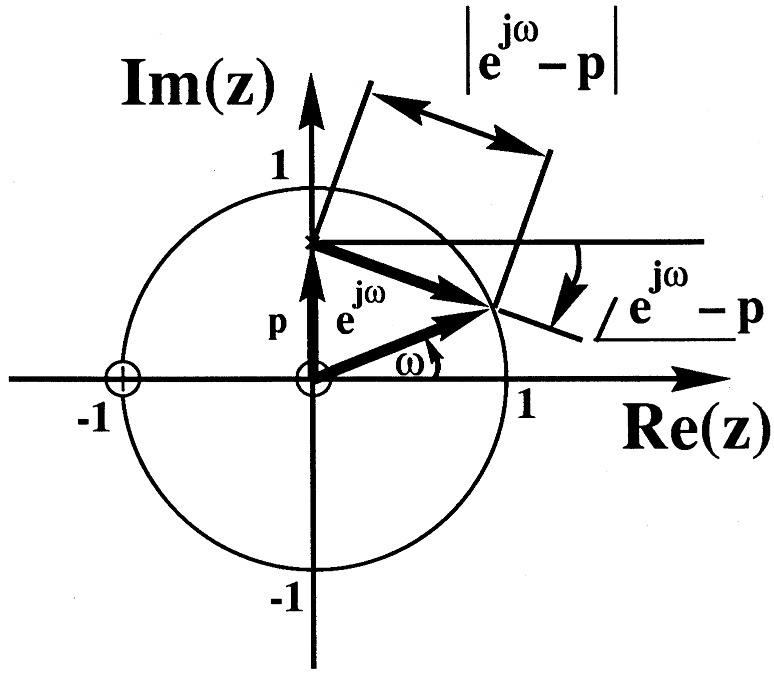
$$H(z) = \frac{z(z+1)}{(z - j/\sqrt{2})(z + j/\sqrt{2})}$$

$$|H(e^{j\omega})| = \frac{|e^{j\omega} + 1|}{|e^{j\omega} - j/\sqrt{2}| \; |e^{j\omega} + j/\sqrt{2}|}$$

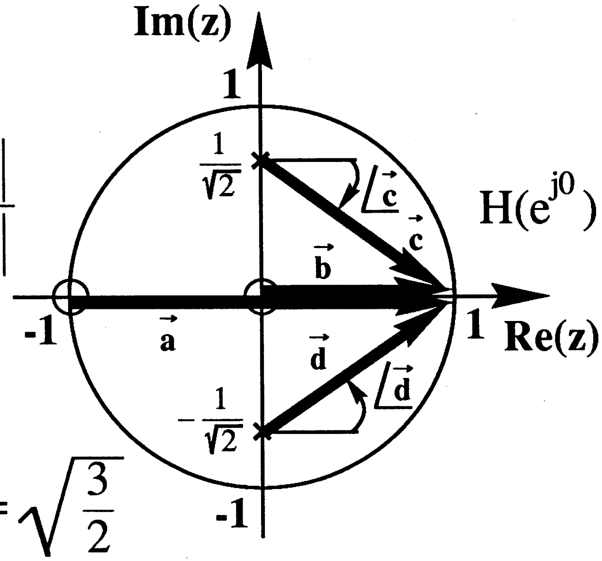
$$\angle H(e^{j\omega}) = \omega + \underline{/e^{j\omega} + 1} - \underline{/e^{j\omega} - j/\sqrt{2}}$$

$$- \underline{/e^{j\omega} + j/\sqrt{2}}$$

Contribution from a single pole



ω = 0



$$\left|H(e^{j0})\right| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}$$

$$|\vec{a}| = 2$$

$$|\vec{b}| = 1$$

$$|\vec{c}| = \sqrt{1+\frac{1}{2}} = \sqrt{\frac{3}{2}}$$

$$|\vec{d}| = \sqrt{\frac{3}{2}}$$

$$\left|H(e^{j0})\right| = \frac{4}{3} = 1.33$$

$$\angle H(e^{j0}) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d}$$

$$\angle\vec{a} = 0$$

$$\angle\vec{b} = 0$$

$$\angle\vec{c} = \arctan\left(\frac{1}{\sqrt{2}}\right)$$

$$\angle\vec{d} = -\angle\vec{c}$$

$$\angle H(e^{j0}) = 0$$

$\omega = \pi / 4$

$$\left|H(e^{j\pi/4})\right| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}$$

$$|\vec{a}| = \sqrt{\left(1 + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

$$= 1.85$$

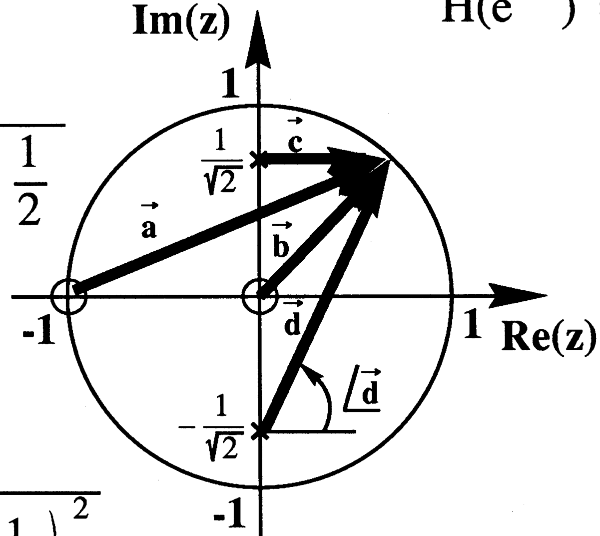
$$|\vec{b}| = 1$$

$$|\vec{c}| = \frac{1}{\sqrt{2}}$$

$$|\vec{d}| = \sqrt{\left(\frac{2}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{5}{2}}$$

$$\left|H(e^{j\pi/4})\right| = 1.65$$



$$H(e^{\pi/4}) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d}$$

$$\angle\vec{a} = 22.5°$$

$$\angle\vec{b} = 45°$$

$$\angle\vec{c} = 0$$

$$\angle\vec{d} = 63.4°$$

$$\angle H(e^{j\pi/4}) = 4.1°$$

$\underline{\omega = \pi / 2}$

$$\left|H(e^{j\pi/2})\right| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}$$

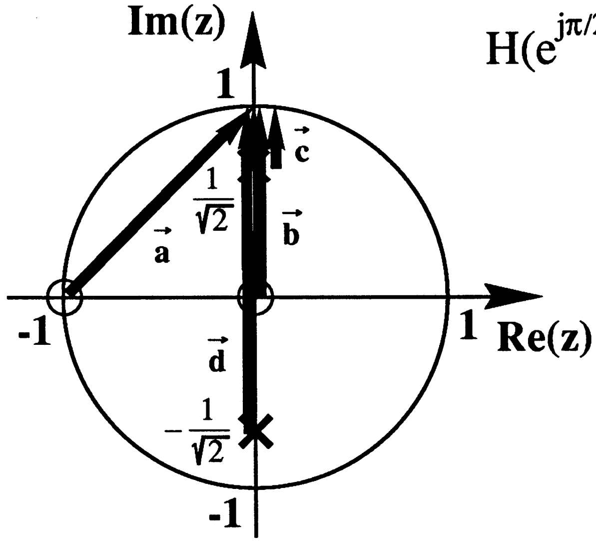
$$|\vec{a}| = \sqrt{2}$$

$$|\vec{b}| = 1$$

$$|\vec{c}| = 1 - \frac{1}{\sqrt{2}}$$

$$|\vec{d}| = 1 + \frac{1}{\sqrt{2}}$$

$$\left|H(e^{j\pi/2})\right| = 2.83$$



$$\angle H(e^{j\pi/2}) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d}$$

$$\angle\vec{a} = 45°$$

$$\angle\vec{b} = 90°$$

$$\angle\vec{c} = 90°$$

$$\angle\vec{d} = 90°$$

$$\angle H(e^{j\pi/2}) = -45°$$

General Rules

- A pole near the unit circle will cause the frequency response to increase in the neighborhood of that pole.
- A zero near the unit circle will cause the frequency response to decrease in the neighborhood of that zero.