Discrete Fourier Transform
The DFT (Chap. 7)

- $X_N(k) = X(\omega)\Big|_{\omega = k\frac{2\pi}{N}}, \; k = 0, 1, \ldots, N-1$

$\Rightarrow$ N equi-spaced "samples" of $X(\omega)$ over $0 \leq \omega < 2\pi$

- whenever you sample in one domain, you get periodic replication in the other domain

- Recall time-domain sampling:

$$x[n] = x_a(t)\Big|_{t=nT_s}$$

$$x_s(t) = x_a(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) \xleftrightarrow{\mathcal{F}} \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(F-k\frac{1}{T_s}\right)$$

$$= \sum_n x_a(nT_s)\,\delta(t-nT_s)$$

- If the sampling rate $F_s = \frac{1}{T_s}$ is not high enough, the periodic replicas in the frequency domain overlap $\Rightarrow$ aliasing
- If $F_s >$ Nyquist rate $(=2W)$, then

$$X_a(t) = \sum_{n=-\infty}^{\infty} \underbrace{x_a(nT_s)}_{x[n]} \frac{\sin\left(\frac{\pi}{T_s}(t-nT_s)\right)}{\frac{\pi}{T_s}(t-nT_s)}$$

- likewise, if we don't sample $X(\omega)$ at enough points in $0<\omega<2\pi$, can have time-domain aliasing

- Foreshadowing, the end result: need to sample $X(\omega)$ at least as much as the length of the signal. That is,

$$N > L$$

no. of points $X(\omega)$ is sampled at over $0<\omega<2\pi$

length of signal.

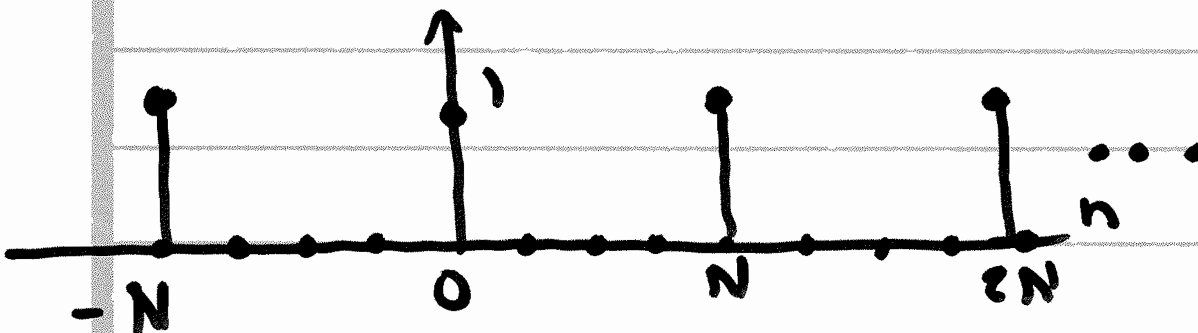
assuming causal signal

$x[n] = 0$ for $n<0$

$= 0$ for $n \geq L$

Mathematical development:

Recall: $$\sum_{k=-\infty}^{\infty} \delta[n-kN] = \underbrace{\frac{1}{N}\sum_{k=0}^{N-1} e^{jk\frac{2\pi}{N}n}}_{\text{Discrete Fourier Series}}$$



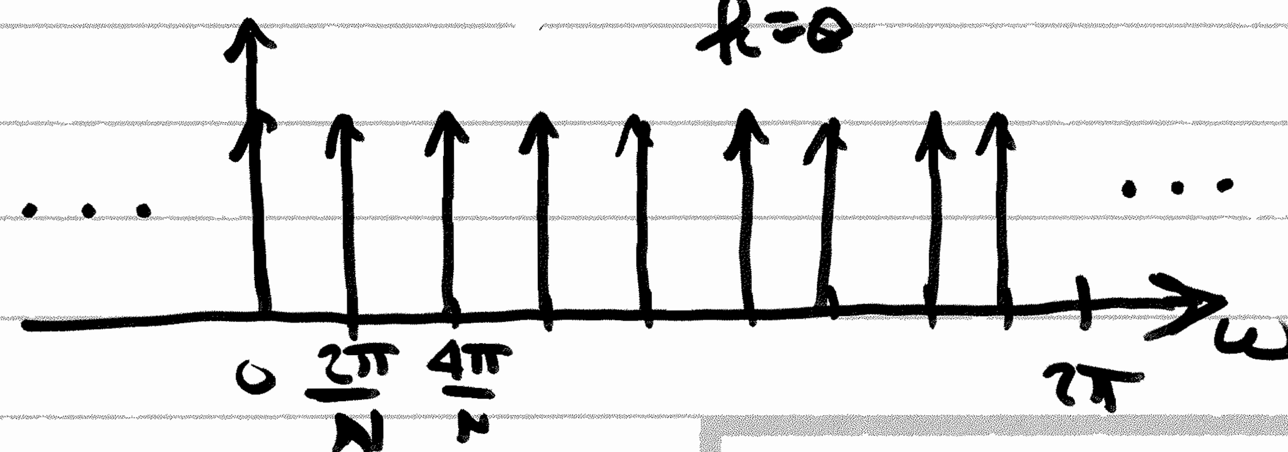
periodic signal

$$\sum_{k=0}^{N-1}\left(e^{j\frac{2\pi}{N}n}\right)^{k} = \frac{1}{N}\,\frac{1-e^{j2\pi n}}{1-e^{j\frac{2\pi}{N}n}}$$

$$= \begin{cases} 1, & n = lN, \ l \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{k=-\infty}^{\infty} \delta[n-kN] = \sum_{k=0}^{N-1} \frac{1}{N} e^{j k \frac{2\pi}{N} n} \xleftrightarrow{\text{DTFT}}$$

$$S'(\omega) = \sum_{k=0}^{N-1} \frac{2\pi}{N} \delta\left(\omega - k\frac{2\pi}{N}\right)$$



Multiplication by $X_s(\omega)$ effectively samples $X(\omega)$ at N equi-spaced points over $0 < \omega < 2\pi$

$$X\ (\omega) = X(\omega)\, S\ (\omega)$$

$$= X(\omega) \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\, \delta\left(\omega - k\frac{2\pi}{N}\right)$$

Multiplication in the frequency domain, implies convolution back in the time domain

$$x[n] * \sum_{k=-\infty}^{\infty} \delta[n-kN] \stackrel{DTFT}{\longleftrightarrow} X(\omega) \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\, \delta\left(\omega - k\frac{2\pi}{N}\right)$$

$$= \left.\sum_{k=-\infty}^{\infty} x[n-kN] \right\} \text{periodic replication in time domain!}$$

$$= x_p[n]$$

- Assume signal "starts" at $n=0$ and is of finite length L : $x[n] \neq 0$ only for $0 \le n \le L-1$

- If $N > L$, then <u>no</u> overlap amongst the periodic replications of $x[n]$

- $$\cdots \underbrace{x[n+N]}_{=0 \text{ for } n>0} + x[n] \; + \underbrace{x[n-N]}_{=0 \text{ for } 0 \le n \le L-1} + \cdots$$

- In this, no time-domain aliasing and $x[n]$ can be isolated as on top of next page

$$x[n] = x_p[n] \; w[n]$$

$$\left\{ \sum_{k=-\infty}^{\infty} x[n-kN] \right\} w[n]$$

where:

$$w[n] = u[n] - u[n-N] \xleftrightarrow{\text{DTFT}} W(\omega) = \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(N-1)}{2}\omega}$$

$$= \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

rectangular window

- Multiplication in time corresponds to (periodic) convolution in frequency

(9)

$$x[n] = x_p[n]\, w[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi} X_p(\omega) \circledast W(\omega)$$

$$X(\omega) = \frac{1}{2\pi}\left(X(\omega) S'(\omega)\right) \circledast W(\omega)$$

$$= \frac{1}{2\pi}\left(X(\omega)\sum_{k=0}^{N-1} \frac{2\pi}{N}\, \delta\left(\omega - k\frac{2\pi}{N}\right)\right) \circledast W(\omega)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X\left(k\frac{2\pi}{N}\right) \delta\left(\omega - k\frac{2\pi}{N}\right) \circledast W(\omega)$$

$$= \sum_{k=0}^{N-1} X_N(k)\, \frac{1}{N}\, W\left(\omega - k\frac{2\pi}{N}\right)$$

$$X(\omega) = \sum_{k=0}^{N-1} X_N(k)\, \frac{\sin\left[\frac{N}{2}\left(\omega - k\frac{2\pi}{N}\right)\right]}{N \sin\left[\frac{1}{2}\left(\omega - k\frac{2\pi}{N}\right)\right]}\, e^{-j\frac{(N-1)}{2}\left(\omega - k\frac{2\pi}{N}\right)}$$

- This is the formula for reconstructing $X(\omega)$ from N samples of $X(\omega)$ equi-spaced over $0 < \omega < 2\pi$ and is ideal as long as $N > L$ where L is the length of $x[n]$