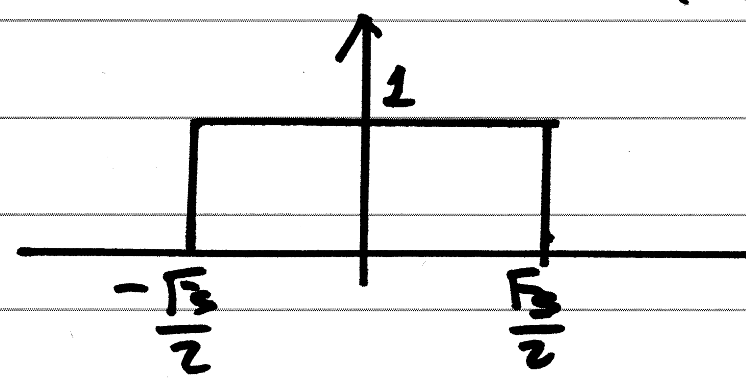
On Digital Filtering to Effect a Fractional Time-Shift

- Consider $x_a(t)$ to be bandlimited such that

  $X_a(f) = 0$ for $|f| > W$

- Ultimately we will sample at $F_s > 2W$ $\left(F_s = \frac{1}{T_s}\right)$

- <u>Note:</u> $\dfrac{\sin\left(\frac{\pi}{T_s} t\right)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$



<u>where:</u> $\frac{F_s}{2} > W$

- <u>Thus:</u> $x_a(t) * \dfrac{\sin\left(\frac{\pi}{T_s} t\right)}{\pi t} = x_a(t)$

$$x_a(t) * \delta(t - t_0) * \sin\left(\frac{\pi}{T_s} t\right) = x_a(t - t_0)$$

$$\Rightarrow x_a(t) * \frac{\sin\left(\frac{\pi}{T_s}(t - t_0)\right)}{\pi(t - t_0)} = x_a(t - t_0)$$

$$x_a(t) * \frac{\sin\left(\frac{\pi}{T_s}(t+t_0)\right)}{\pi(t+t_0)} = x_a(t+t_0)$$

- consider $t_0 = \frac{\ell}{L} T_s$:

$$x_a(t) * \frac{\sin\left(\frac{\pi}{T_s}(t+\frac{\ell}{L}T_s)\right)}{\pi(t+\frac{\ell}{L}T_s)} = x_a\left(t+\ell\frac{T_s}{L}\right)$$

- Since both $x_a(t)$ and the ideal lowpass interpolating filter are bandlimited to $F_s/2$, we can sample eqn. above at $t = nT_s$ to obtain:

$$x_a(nT_s) \underset{\substack{\uparrow \\ \text{DT convolution}}}{*} \frac{\sin\left(\frac{\pi}{T_s}(nT_s+\frac{\ell}{L}T_s)\right)}{\pi(nT_s+\frac{\ell}{L}T_s)} = x_a\left(nT_s+\ell\frac{T_s}{L}\right)$$

$$x[n] * \frac{\sin\left(\pi(n+\frac{\ell}{L})\right)}{\pi(n+\frac{\ell}{L})} = x_a\left(t+\frac{\ell}{L}T_s\right)\Big|_{t=nT_s}$$