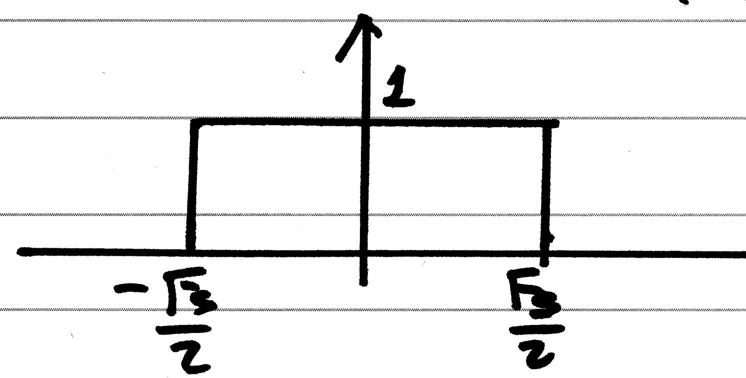
On Digital Filtering to Effect a Fractional Time-Shift

- Consider $x_a(t)$ to be bandlimited such that $X_a(f) = 0$ for $|f| > W$
- Ultimately we will sample at $F_s > 2W$ $\left(F_s = \frac{1}{T_s}\right)$
- <u>Note:</u> $\dfrac{\sin\left(\frac{\pi}{T_s} t\right)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$



<u>where:</u> $\frac{F_s}{2} > W$

- <u>Thus:</u> $x_a(t) * \dfrac{\sin\left(\frac{\pi}{T_s} t\right)}{\pi t} = x_a(t)$

$$x_a(t) * \delta(t - t_0) * \frac{\sin\left(\frac{\pi}{T_s} t\right)}{\pi t} = x_a(t - t_0)$$

$$\Rightarrow x_a(t) * \frac{\sin\left(\frac{\pi}{T_s}(t - t_0)\right)}{\pi (t - t_0)} = x_a(t - t_0)$$

$$x_a(t) * \frac{\sin\left(\frac{\pi}{T_s}(t+t_0)\right)}{\pi(t+t_0)} = x_a(t+t_0)$$

- consider $t_0 = \frac{\ell}{L} T_s$:

$$x_a(t) * \frac{\sin\left(\frac{\pi}{T_s}\left(t+\frac{\ell}{L}T_s\right)\right)}{\pi\left(t+\frac{\ell}{L}T_s\right)} = x_a\left(t+\ell\frac{T_s}{L}\right)$$

- Since both $x_a(t)$ and the ideal lowpass interpolating filter are bandlimited to $F_s/2$, we can sample eqn. above at $t = nT_s$ to obtain:

$$x_a(nT_s) \underset{\substack{\uparrow \\ \text{DT convolution}}}{*} \frac{\sin\left(\frac{\pi}{T_s}\left(nT_s+\frac{\ell}{L}T_s\right)\right)}{\pi\left(nT_s+\frac{\ell}{L}T_s\right)} \; T_s = \Delta t \; = x_a\left(nT_s+\ell\frac{T_s}{L}\right)$$

$$x[n] * \frac{\sin\left(\pi\left(n+\frac{\ell}{L}\right)\right)}{\pi\left(n+\frac{\ell}{L}\right)} = x_a\left(t+\frac{\ell}{L}T_s\right)\Big|_{t=nT_s}$$

Recall relationship between CTFT and DTFT if Sampling Rate is above (or at) Nyquist Rate

$$X(\omega) = F_s \, X_a\left(\frac{F_s}{2\pi}\omega\right) \quad \text{for } -\pi < \omega < \pi$$

$$h_{a\ell}(t) = \frac{\sin\left(\frac{\pi}{T_s}\left(t + \frac{\ell}{L}T_s\right)\right)}{\pi\left(t + \frac{\ell}{L}T_s\right)} \longleftrightarrow e^{j2\pi f \frac{\ell}{L} T_s} = H_{a\ell}(f)$$

$$\text{for } -\frac{F_s}{2} < f < \frac{F_s}{2}$$

Thus:

$$H_\ell(\omega) = T_s F_s \, e^{j2\pi \frac{F_s}{2\pi}\omega \frac{\ell}{L} T_s} = e^{j\frac{\ell}{L}\omega} \quad \text{for } -\pi < \omega < \pi$$

$\Delta t = T_s$ in DT convolution approx. to CT convolution integral

Approximating area under curve via sum of area under rectangles:

$$\int_{-\infty}^{\infty} f(x)\,dx = \sum_{k=-\infty}^{\infty} \underbrace{f(k\Delta x)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}$$

Applied to convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \sum_{k=-\infty}^{\infty} x(k\Delta\tau)\,h(t-k\Delta\tau)\,\Delta\tau$$

Then discretize t (or sample $y(t)$):

$$y[n] = y(nT_s) = y(n\Delta\tau)$$

$$y[n] = \sum_{k} x(k\Delta\tau)\,h((n-k)\Delta\tau)\,\Delta\tau$$

$$= \sum_{k} x[k]\,h[n-k]\,\Delta\tau$$

where:

$x[k] = x(k\Delta\tau)$

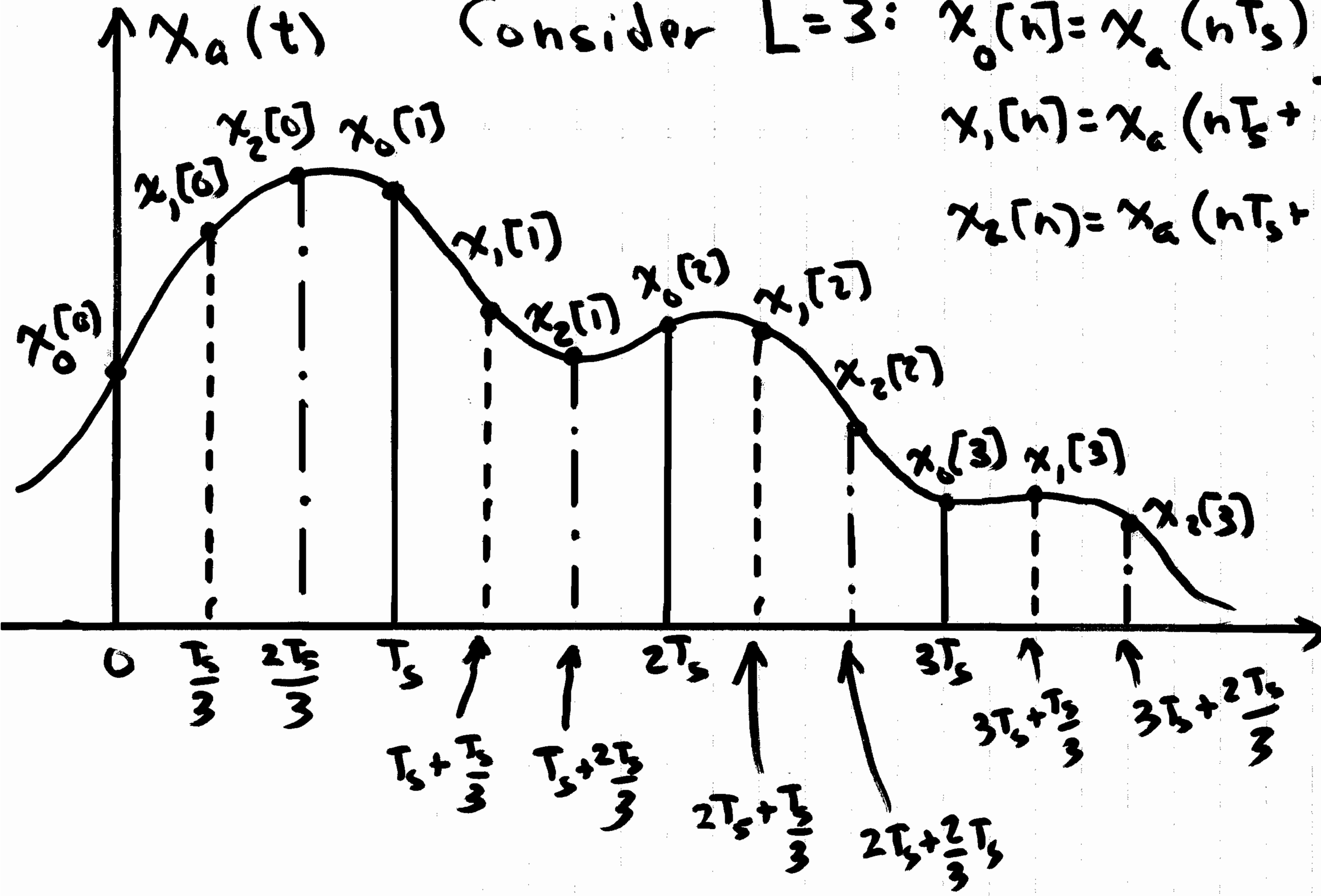
$h[k] = h(k\Delta\tau)$

Consider $L=3$:

$$x_0[n] = x_a(nT_s)$$

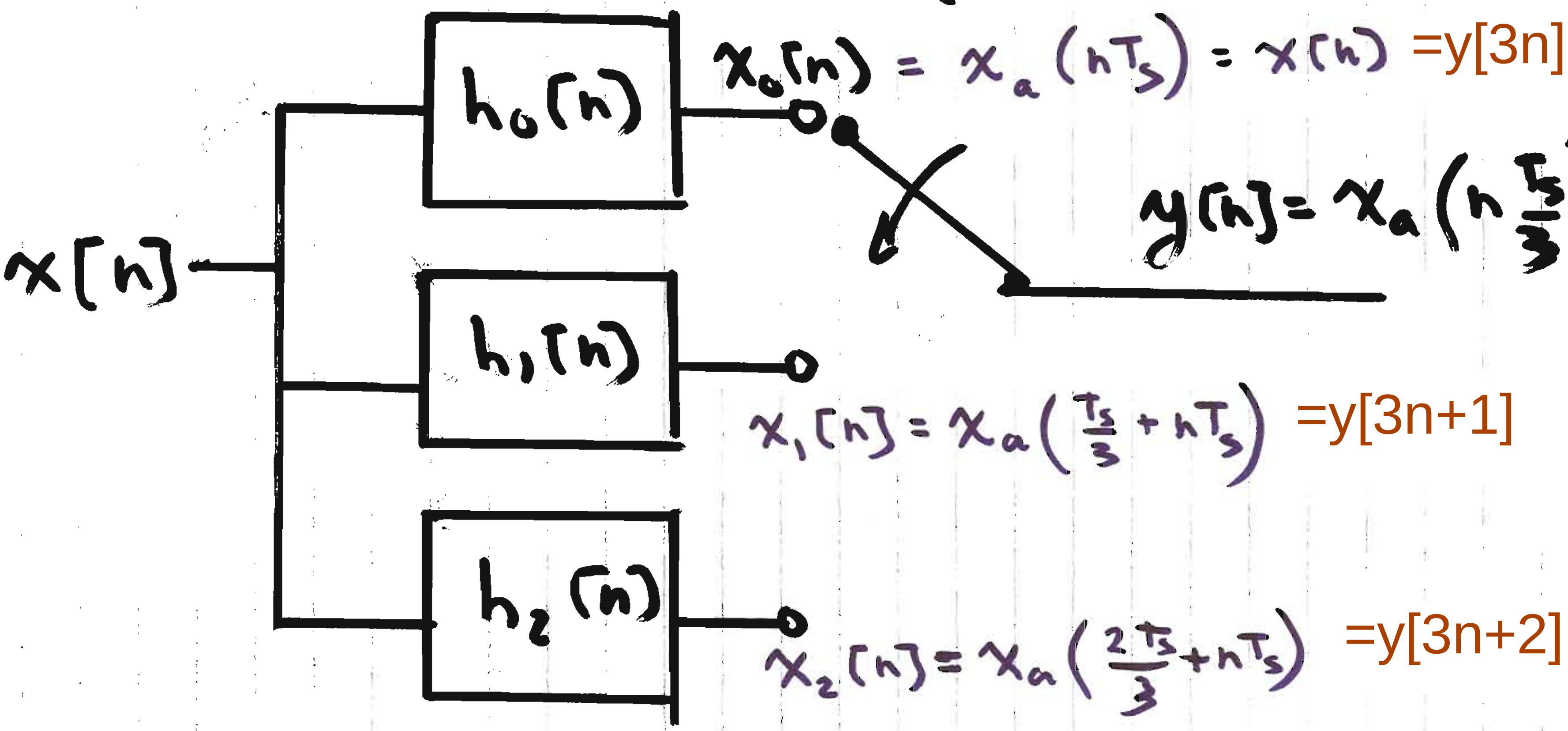
$$x_1[n] = x_a\left(nT_s + \frac{T_s}{3}\right)$$

$$x_2[n] = x_a\left(nT_s + 2\frac{T_s}{3}\right)$$



$$h[n] = \frac{3\sin\left(\frac{\pi}{3}n\right)}{\pi n}$$

$$h_0[n] = h[Ln]$$
$$h_1[n] = h[Ln+1]$$
$$h_2[n] = h[Ln+2]$$

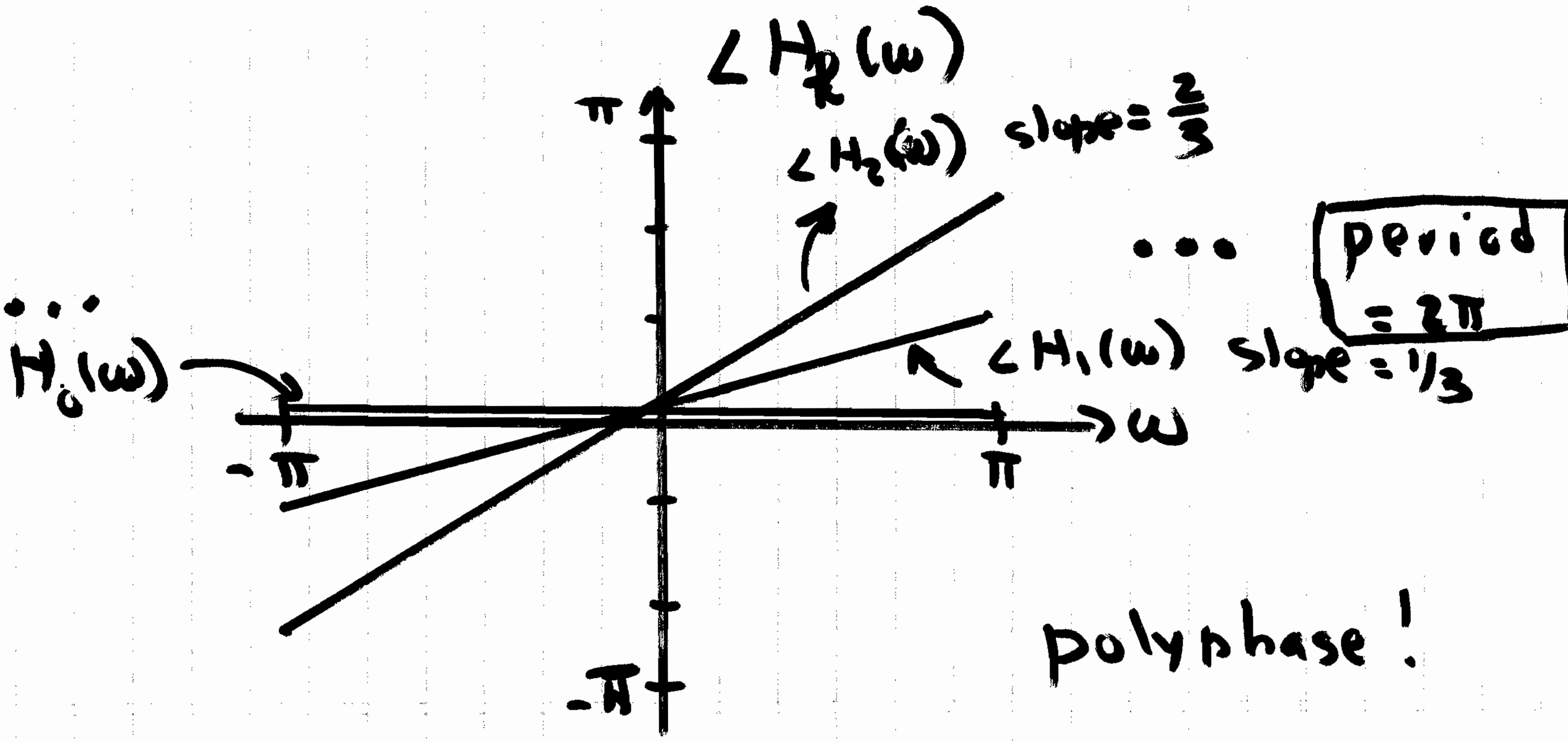


In this set of notes, x_0 [n], x_1 [n] and x_2 [n] represent the fractional time-shift filter outputs in the case of upsampling ONE signal.

for $|\omega| < \pi$: $H_0(\omega) = 1$

$H_1(\omega) = e^{j\frac{1}{3}\omega} \Rightarrow \angle H_1(\omega) = \frac{\omega}{3}$

$H_2(\omega) = e^{j\frac{2}{3}\omega} \Rightarrow \angle H_2(\omega) = \frac{2}{3}\omega$



polyphase !