Impulse response for simple first-order

Difference Equations

\[ y[n] = a \cdot y[n-1] + x[n] \]

1. Determine impulse response. (Zero initial conditions)
2. When \( x[n] = \delta[n] \), \( y[n] = h[n] \)
3. \( h[n] = 0 \) for \( n < 0 \) → causal
4. \( h[n] = a \cdot h[n-1] + \delta[n] \) \( \text{iterate through recursively} \)
5. \( h[0] = a \cdot h[-1] + 1 = 1 = a^0 \)
6. \( h[1] = a \cdot h[0] = a \cdot 1 = a \)
7. \( h[2] = a \cdot h[1] = a \cdot a = a^2 \)
8. \( h[n] = a \cdot h[n-1] \Rightarrow a^n = a \cdot a^{n-1} \)
For $a = 1 \Rightarrow y[n] = y[n-1] + x[n]$

- impulse response is unit step

$$h[n] = (-1)^n u[n] = u[n]$$

This is the same impulse response as for the system:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- Two different realizations of the same system

- And recall (side-note) that the inverse system has the difference equation $y[n] = x[n] - x[n-1]$ since the impulse response is $h[n] = \delta[n] - \delta[n-1]$ and:

  $$u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n]$$
**Second Example**

\[ y(n) = ay(n-1) + x(n) - a^D x(n-D) \]

- Iterate recursively to find impulse response
- \( h(n) \neq 0 \) for \( n < 0 \) \( \implies \) causal

\[ h(n) = ah(n-1) + \delta(n) - a^D \delta(n-D) \]

\[ h(0) = ah(-1) + 1 + 0 = 1 \]

\[ h(1) = ah(0) + 0 + 0 = a \]

\[ \vdots \]

\[ h(D-1) = a^{D-1} + 0 + 0 = a \]

\[ h(D) = ah(D-1) - a^D \delta(n) = a a^{D-1} - a^D = a^D - a^D = 0 \]

\[ h(D+1) = ah(D) + 0 + 0 = 0 \]

\[ \Rightarrow h(n) = a^n \{ u(n) - u(n-D) \} \]
• What is output when $x(n) = b^\{u(n) - u(n-N)\}$?

• Have multiple options for computing convolution of two finite-length geometric sequences

\[ y(n) = \{a^n (u(n) - u(n-D))\} * \{b^n (u(n) - u(n-N))\} \]

when given specific values for $a, b, D > N$

• Example: $y(n) = \frac{1}{4} y(n-1) + x(n) = \left(\frac{1}{64}\right)x(n-3)$

Find output when:

$X[n] = 4 \times 4^n \{u(n) - u(n-3)\} = 4^{n+3} \{u(n) - u(n-3)\}$

• recall homogeneity aspect to linearity

• ultimately need to convolve two length=3 sequences

\[ \{4, 16, 64\} \ast \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\} \]

\[ n=0 \quad n=0 \]
\[ n \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

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<th>16 \ h(n-1)</th>
<th>64 \ h(n-2)</th>
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Length \( h(n) \) = \{ 4, 17, 68.25, 17, 4 \} \quad \text{Length} = 3 + 3 - 1

\[ y(n) = \begin{cases} 4 & \text{if } n = 0 \\ 17 & \text{if } n = 1 \\ 68.25 & \text{if } n = 2 \\ 17 & \text{if } n = 3 \\ 4 & \text{if } n = 4 \\ 0 & \text{if } n = 5 \end{cases} \]