Problem 1. An OFDM signal is synthesized as a sum of four sinewaves of length 4 as

$$\tilde{x}[n] = \frac{1}{4}\sum_{k=0}^{3} b_k e^{j2\pi\frac{k}{4}n}\{u[n] - u[n-4]\}$$

Each of the four values b_k is one of the eight values listed in the symbol alphabet below.

$$b_k \in \{-7, -5, -3, -1, 1, 3, 5, 7\} \quad k = 0, 1, 2, 3$$

We know in advance that the signal we transmit will be convolved with a filter of length $L = 3$. Thus, we add a cyclic prefix of length 3 which effective creates a sum of sinewaves of length 7 as prescribed below:

$$x[n] = \frac{1}{4}\sum_{k=0}^{3} b_k e^{j2\pi\frac{k}{4}n}\{u[n+3] - u[n-4]\}$$

The sequence of length 7 above is convolved with the filter below of length 3.

$$h[n] = \{2, 4, 1\} = 2\delta[n] + 4\delta[n-1] + \delta[n-2]$$

This ultimately yields the following sequence of length 7+3-1 = 9

$$y[n] = x[n] * h[n] = \{\cancel{2+2j}, \cancel{6+4j}, \cancel{7-j}, 13-4j, 19+j, 10+4j, 7-j, \cancel{5-4j}, \cancel{1-j}\}$$

Your task is to determine the numerical values of each of the four symbols: b_0, b_1, b_2, and b_3. Explain and show all the steps in determining your answers. Please lay your work out nicely with logical ordering. Your work will be more important than your final answers.

Consider $y[n]$. We ignore L beggining values & $L-1$ end values.

$L = 3$.

$\therefore \quad \tilde{y}[n] = \{13-4j, \ 19+j, \ 10+4j, \ 7-j\}$

$h[n] = \{2, 4, 1\} \quad \therefore \quad H(z) = 2 + 4z^{-1} + z^{-2}$

For $H_N(k)$, $\omega = \dfrac{2\pi k}{4} = \dfrac{\pi k}{2}$ $\quad k = 0, 1, 2, 3$

$\omega = 0$, $\quad z = 1$ $\qquad \omega = \pi$, $\quad z = -1$

$\omega = \pi/2$ $\quad z = e^{j\pi/2} = j$ $\qquad \omega = \frac{3\pi}{2}$, $\quad z = -j$

$\therefore \; H_4(0) = 7$

$$H_4(1) = 2 + \frac{4}{j} + \frac{1}{j^2} = 1 - 4j$$

$$H_4(2) = 2 - 4 + 1 = -1$$

$$H_4(3) = 2 + \frac{4}{(-j)} + \frac{1}{(-j)^2} = 1 + 4j$$

Now, we will need the sinewave values,

$$e^{j\frac{2\pi kn}{4}}(u(n) - u(n-4)) = S_k[n]$$

$$k = 0 \rightarrow S_0[n] = \{1, 1, 1, 1\}$$

$$k = 1 \rightarrow S_1[n] = e^{j\frac{\pi n}{2}} = \{1, j, -1, -j\}$$

$$k = 2, \rightarrow S_2[n] = e^{j\pi n} = \{1, -1, 1, -1\}$$

$$k = 3, \rightarrow S_3[n] = e^{j\frac{3\pi n}{2}} = \{1, -j, -1, j\}$$

Now, use:
$$\sum_{n=0}^{3} \tilde{y}(n) S_k^*(n) = b_k H_4(k)$$

$$\therefore \; k = 0, \quad 7b_0 = ? \qquad y[n] = \{13 - 4j, 19 + j, 10 + 4j, 7 - j\}$$

$$\therefore \; b_0 = \frac{13 - \cancel{4j} + 19 + \cancel{j} + 10 + \cancel{4j} + 7 - \cancel{j}}{7} = 7$$

$$b_1(1 - 4j) = 13 - 4j - 19j + 1 - 10 - 4j + 7j + 1$$

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$\therefore \quad b_1 = \underline{\underline{5}}$

$\underline{\underline{k=2}}, \quad (-1)\, b_2 = 13 - \cancel{4j} - 19 - \cancel{j} + 10 + \cancel{4j} - 7 + \cancel{j}$

$\therefore \quad b_2 = \underline{\underline{3}}$

$\underline{\underline{k=3}}, \quad (1+4j)\, b_3 = 13 - 4j + 19j - 1 - 10 - 4j - 7j - 1$

$\therefore (1+4j)\, b_3 = 1 + 4j$

$\therefore \quad b_3 = \underline{\underline{1}}$

Thus,

$$\boxed{\begin{aligned} b_0 &= 7 \\ b_1 &= 5 \\ b_2 &= 3 \\ b_3 &= 1 \end{aligned}}$$

(Ans.)

Problem 2. Problem 1 is modified as follows. The only thing that is changed is that the filter is changed to (only the $n = 2$ value is changed)

$$h[n] = \{2, 4, 2\} = 2\delta[n] + 4\delta[n-1] + 2\delta[n-2]$$

Note that $h[n]$ may also be expressed as two times the convolution of the sequence $\{1, 1\}$ with itself as

$$h[n] = \{2, 4, 2\} = 2\{1, 1\} * \{1, 1\}$$

Everything else is the same. We send the same four symbols values b_k as in Problem 1. We add a cyclic prefix of length 3 which effective creates a sum of sinewaves of length 7 as below, again, the same as in Problem 1.

$$x[n] = \frac{1}{4}\sum_{k=0}^{3} b_k e^{j2\pi\frac{k}{4}n}\{u[n+3] - u[n-4]\}$$

The sequence of length 7 above is convolved with the filter below of length 3

$$h[n] = \{2, 4, 2\} = 2\delta[n] + 4\delta[n-1] + 2\delta[n-2]$$

This ultimately yields the following sequence of length 7+3-1 = 9

$$y[n] = x[n] * h[n] = \{2 + 2j, 6 + 4j, 8, 14 - 4j, 20, 14 + 4j, 8, 6 - 4j, 2 - 2j\}$$

You do NOT have to determine the values of b_k, $k = 0, 1, 2, 3$. In fact, given this new filter and given ONLY the values of $y[n]$ listed above, you will NOT be able to determine one of the values of b_k, $k = 0, 1, 2, 3$.

TASK: Determine which one of the four values b_0, b_1, b_2, and b_3 you are unable to determine, and explain why.

$$H_4(z) = 2 + 4z^{-1} + 2z^{-2}$$

$$H_4(0) = 4 + 2 + 2 = 8$$

$$H_4(1) = 2 + \frac{4}{j} + \frac{2}{j^2} = -4j$$

$$H_4(2) = 2 - 4 + 2 = 0 \,! \leftarrow$$

$$H_4(3) = +4j \quad (\because H(k) = H^*(N-k))$$

Now

Since,
$$b_k = \frac{\sum_{n=0}^{3} y[n]\, s_k^*[n]}{H_N(k)}$$

b_2 cannot be determined!
[0 in the denominator]

Problem 3. [30 points] Consider a single-pole, analog all-pass filter with transfer function (Laplace Transform of impulse response)

$$H_a(s) = \frac{s + p_a^*}{s - p_a}$$

where the subscript a denotes analog. If one substitutes $s = j\Omega$ to get the frequency response (Fourier Transform of impulse response)

$$H_a(\Omega) = \frac{j\Omega + p_a^*}{j\Omega - p_a}$$

it is easy to show that the magnitude $|H_a(\Omega)| = 1$ for all Ω.

(a) The transfer function for a digital filter is obtained via the bilinear transform

$$s = \frac{z-1}{z+1}$$

We discussed three properties of the bilinear transform in class. Which property of the bilinear transform guarantees us that an all-pass analog filter will be transformed into an all-pass digital filter?

(b) Consider the case where the pole is located at $p_a = -0.2 + 0.4j$. Determine the values of the pole, p_d, and zero, z_d, of the digital filter obtained via the bilinear transform.

(i) Draw a pole-zero diagram for the resulting digital filter.

(ii) Is the resulting digital filter stable? Explain your answer.

(iii) Is the resulting digital filter an all-pass filter? Explain your answer.

(iv) What digital frequency is the analog frequency $\Omega = \sqrt{3}$ (in radians) mapped to?

(v) Determine and write the difference equation for the resulting digital filter.

b) pole at p_a. Zero at $-p_a^*$ → for the analog filter.

$s = -0.2 + 0.4j$

$\therefore \; z = \frac{1+s}{1-s}$ $\Rightarrow z = \frac{0.8 + 0.4j}{1.2 - 0.4j}$

$z = \frac{2+j}{3-j}$

$z = \frac{(2+j)(3+j)}{10} = \frac{1}{2} + \frac{1}{2}j$

9

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$$\therefore \frac{Y(z)}{X(z)} = \frac{1-(1+j)z^{-1}}{1-(0.5+0.5j)z^{-1}}$$

$$\therefore \boxed{y(n) = x(n) - (1+j)\,x(n-1) + \left(\frac{1+j}{2}\right) y(n-1)} \quad \text{(Ans.)}$$

a) Since the $j\Omega$ imaginary axis gets mapped to the unit circle.

one-to-one

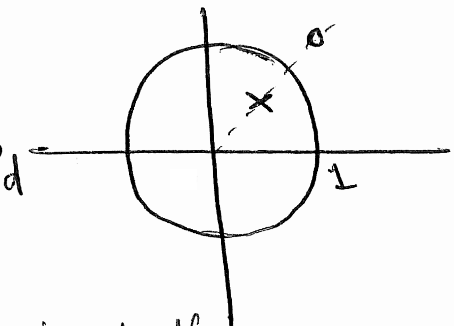
$-p_a^*$ is $= -[-0.2 - 0.4j]$

$= 0.2 + 0.4j$

$$z = \frac{1.2 + 0.4j}{0.8 - 0.4j} = \frac{3 + j}{2 - j} = \frac{5 + 5j}{5} = 1 + j$$

Thus, zero at $1 + j = z_d$

pole at $\frac{1}{2} + \frac{1}{2}j = p_d$



(ii) The resulting digital filter is stable as pole is inside unit circle.

(iii) Resulting filter is indeed all pass as,

$$\text{pole} = \frac{1}{(\text{zero})^*}$$

(iv) $\Omega = \tan(\omega/2)$

$\therefore \Omega = \sqrt{3} \Rightarrow \omega = 2\tan^{-1}(\sqrt{3})$

$= \frac{2\pi}{3}$

(v) Resulting digital filter,

$$H(z) = \frac{z - (1 + j)}{z - (0.5 + 0.5j)}$$

Problem 4. For all parts of this problem, $x[n]$ is the fine-length sinewave of length $L = 8$ with frequency $\omega_o = \pi/2$ defined below, and $h[n]$ is a causal filter of length $M = 4$ which may be expressed in sequence form as $h[n] = \{1, 2, -2, -1\}$.

$$x[n] = \cos\left(\frac{\pi}{2}n\right)\{u[n] - u[n-8]\} \qquad\qquad h[n] = \{1, 2, -2, -1\}$$

(a) Compute the linear convolution of $x[n]$ and $h[n]$. Indicate which points are the transient points (partial overlap) at the beginning and end, and also which points are "pure" sinewave (full overlap.)

(b) With $X_N(k)$ computed as the 8-pt DFT of $x[n]$ and $H_N(k)$ computed as the 8-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is formed. Determine the $N = 8$ values of the 8-pt Inverse DFT of $Y_N(k) = X_N(k)H_N(k)$.

(c) Using your answer to (a), explain your answer to (b) by mathematically illustrating the time-domain aliasing effect.

(d) The product sequence $Y_N(k) = X_N(k)H_N(k)$, formed as directly above with $N = 8$, is used in Eqn (1). Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$, computed according to Eqn (1) below:

$$Y_r(\omega) = \sum_{k=0}^{N-1} Y_N(k) \frac{\sin\left[\frac{N}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]}{N \sin\left[\frac{1}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]} e^{-j\frac{N-1}{2}\left(\omega - \frac{2\pi k}{N}\right)}$$

$\frac{\cancel{2}\pi k}{8}$ (1

a) $x[n] \rightarrow \{1, 0, -1, 0, 1, 0, -1, 0\}$ $e^{-j\frac{7}{2}}$

$\sum x(k)h(n-k) = \sum_{k=0}^{3} h(k)x(n-k)$

1	0	−1	0	1	0	−1	0			
	2	0	−2	0	2	0	−2	0		
		−2	0	2	0	−2	0	2	0	
			−1	0	1	0	−1	0	1	0

$\{1, 2, -3, -3, 3, 3, -3, -3, 2, 1, 0\}$

(pure sine)

3 transient point at beg. & end.

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$$H_8(6) = 1 + 2e^{-j\frac{3\pi}{2}} - 2e^{-j3\pi} - 1 \cdot e^{-j\frac{9\pi}{2}}$$

$$k=6,\ \omega = \frac{3\pi}{2} \qquad = 1 + 2j + 2 + j = 3 + 3j$$

$$\therefore Y_8(k) = \frac{(3-3j)}{2} \cdot 8\,\delta(k-2) + \frac{(3+3j)}{2}\, 8\,\delta(k-6)$$

$$\therefore y_p[n] = \left(\frac{3-3j}{2}\right) e^{j\frac{\pi n}{2}} + \frac{(3+3j)}{2} e^{j\frac{3\pi n}{2}}$$

$$= \{3, 3, -3, -3, 3, 3, -3, -3\}$$

$$\therefore Y_8(\omega) = \frac{(3-3j)}{2} \frac{\sin\left(4(\omega - \pi/2)\right)}{\sin\left(\frac{1}{2}(\omega - \frac{\pi}{2})\right)} e^{-j\frac{7}{2}(\omega - \frac{\pi}{2})}$$

$$+ \frac{(3+3j)}{2} \frac{\sin\left(4(\omega - 3\pi/2)\right)}{\sin\left(\frac{1}{2}(\omega - \frac{3\pi}{2})\right)} e^{-j\frac{7}{2}(\omega - \frac{3\pi}{2})}$$

(Ans:)

(b) Call the 8 point inv DFT as,

$y_8[n]$.

Last 3 points aliased into first 3,

$\therefore \quad y_8[n] = \{3, 3, -3, -3, 3, 3, -3, -3\}$

Now, $\quad x(n) = \dfrac{e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}}{2} \quad 0 \le n \le 7$

$\dfrac{\pi}{2} = \dfrac{2\pi}{8}(2) \qquad \underline{N = 8}$

$\therefore \quad X_8(k) = \dfrac{8}{2}\delta(k-2) + \dfrac{8}{2}\delta(k-6)$

$X_8(k) = 4 \cdot [\delta(k-2) + \delta(k-6)]$

$\therefore \quad Y_8(k) = X_8(k) H_8(k)$

$= 4 H_8(2)\delta(k-2) + 4 H_8(6)\delta(k-6)$

$H(\omega) = 1 + 2e^{-j\omega} - 2e^{-2j\omega} - e^{-3j\omega}$

$\therefore \quad \omega = \dfrac{2\pi k}{8} = \dfrac{\pi k}{4} \quad \therefore k = 2 \Rightarrow \omega = \dfrac{\pi}{2}$

$\therefore H_8(2) = 1 + 2(-j) + 2 + j$

$= 3 - 3j$

$$\therefore \quad e^{j\frac{3\pi}{2}n} = e^{-j\frac{\pi n}{2}}$$

$$\therefore \quad y_p[n] = \left(\frac{3-3j}{2}\right) e^{j\frac{\pi n}{2}} + \frac{(3+3j)}{2} e^{-j\frac{\pi n}{2}}$$

$$= 3\cos\frac{\pi n}{2} + 3\sin\frac{\pi n}{2}$$

$$= \{3, 3, -3, -3, 3, 3, -3, -3\}$$

Thus, there was time domain aliasing!

$$y_p[n] = y[n] + y[n+8],$$

$y(n) \rightarrow$ 1, 2, −3, −3, 3, +3, −3, −3, 2, 1, 0

$y(n+8) \rightarrow$ . . . 2 1 0

$\{3, 3, -3, -3, 3, 3, -3, -3\}$

Problem 5. Consider the upsampler system below in Figure 1.

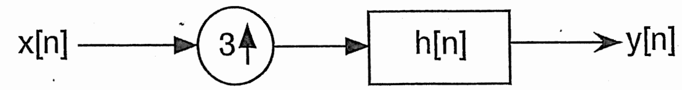


Figure 1.

(a) Draw block diagram of efficient implementation of the upsampler system in Fig. 1.

(b) Your answer to part (a) should involve the polyphase components of $h[n]$: $h_0[n] = h[3n]$, $h_1[n] = h[3n+1]$, and $h_2[n] = h[3n+2]$ and the DTFT of $h[n]$, denoted $H(\omega)$.

(c) Consider that the input to the system in Figure 1 is a sampled version of the analog signal in Figure 2. For the remaining parts of this problem, the input signal is as defined below where $x_a(t)$ is the analog signal in Figure 2. Assume that $1/T_s = 1$ is above the Nyquist rate for this signal. That is, even though this signal is not strictly bandlimited, assume that aliasing effects are negligible.

$$x[n] = x_a(nT_s), \qquad T_s = 1$$

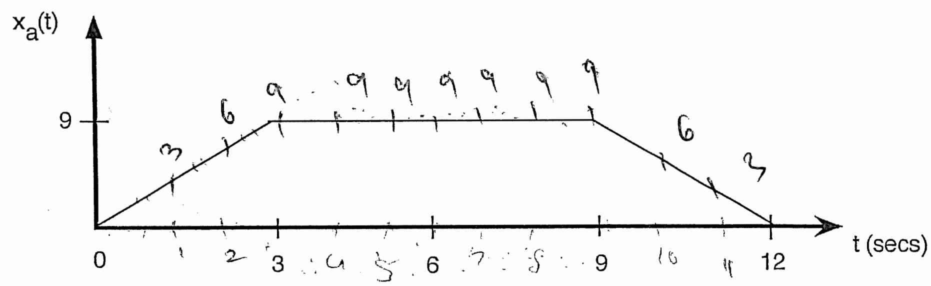


Figure 2.

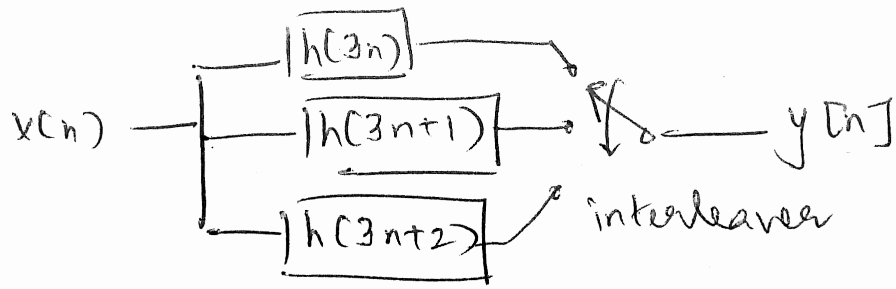
(i) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y[n]$ of the system in Figure 1, when $x[n]$ is input to the system. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

(ii) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_0[n] = x[n] * h_0[n]$, when $x[n]$ is input to the filter $h_0[n] = h[3n]$. Write output in sequence form (indicating where is $n = 0$ OR do stem plot.

(iii) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_1[n] = x[n] * h_1[n]$, when $x[n]$ is input to the filter $h_1[n] = h[3n+1]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

(iv) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_2[n] = x[n] * h_2[n]$, when $x[n]$ is input to the filter $h_2[n] = h[3n+2]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

a) Efficient implementation is :-



$x[n] = \{\overset{\downarrow}{0}, 3, 6, 9, 9, 9, 9, 9, 9, 9, 6, 3, 0\}$

(i) output
$= \{\overset{\downarrow}{0}, 1, 2, 3, \cancel{\;}\; 4, 5, 6, 7, 8, 9, 9, 9, 9, 9, 9, 9, 9,$
$9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 8, 7, 6, 5, 4, 3, 2, 1,$
$0\}$

(ii) output $= \{\overset{\downarrow}{0}, 3, 6, 9, 9, 9, 9, 9, 9, 9, 6, 3, 0\}$

(iii) output $= \{\overset{\downarrow}{1}, 4, 7, 9, 9, 9, 9, 9, 9, 8, 5, 2, 0\}$

(iv) output $= \{\overset{\downarrow}{2}, 5, 8, 9, 9, 9, 9, 9, 9, 7, 4, 1, 0\}$

(fractional time shifts)