

SOLUTION

ECE 538 Digital Signal Processing I Final Exam 2014
Test Date: Tuesday, 16 December 2014

Cover Sheet

Test Duration: 120 minutes.

Open Book but Closed Notes.

One 8.5 x 11 handwritten crib sheet allowed.

Calculators NOT allowed.

This test contains **FOUR** problems.

All work should be done in the proper space provided.

Good luck in this final exam!
Great having you in class this semester!
Have a joyous holiday season!

Problem 1. Consider the upsampler system below in Figure 1.

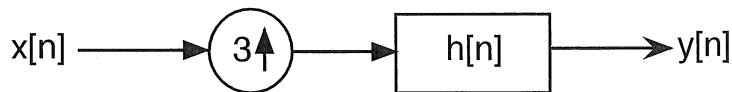


Figure 1.

- (a) Draw block diagram of efficient implementation of the upsampler system in Fig. 1.
- (b) Your answer to part (a) should involve the polyphase components of $h[n]$: $h_0[n] = h[3n]$, $h_1[n] = h[3n + 1]$, and $h_2[n] = h[3n + 2]$ and the DTFT of $h[n]$, denoted $H(\omega)$.
- (i) For the ideal case where $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$, plot the magnitude of the DTFT of $h_0[n] = h[3n]$, $H_0(\omega)$, over $-\pi < \omega < \pi$.
- (ii) For the general case where $h[n]$ is an arbitrary impulse response, express the DTFT of $h_1[n] = h[3n + 1]$, denoted $H_1(\omega)$, in terms of $H(\omega)$.
- (iii) For the ideal case where $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$, plot both the magnitude AND phase (two separate plots) of the DTFT $h_1[n] = h[3n + 1]$, $H_1(\omega)$, over $-\pi < \omega < \pi$.
- (iv) Express the DTFT of $h_2[n] = h[3n + 2]$, denoted $H_2(\omega)$, in terms of $H(\omega)$.
- (v) For the ideal case where $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$, plot both the magnitude AND phase (two separate plots) of the DTFT $h_2[n] = h[3n + 2]$, $H_2(\omega)$, over $-\pi < \omega < \pi$.
- (c) Consider that the input to the system in Figure 1 is a sampled version of the analog signal in Figure 2. For the remaining parts of this problem, the input signal is as defined below where $x_a(t)$ is the analog signal in Figure 2. Assume that $1/T_s = 1/3$ is above the Nyquist rate for this signal. That is, even though this signal is not strictly bandlimited, assume that aliasing effects are negligible.

$$x[n] = x_a(nT_s), \quad T_s = 3$$

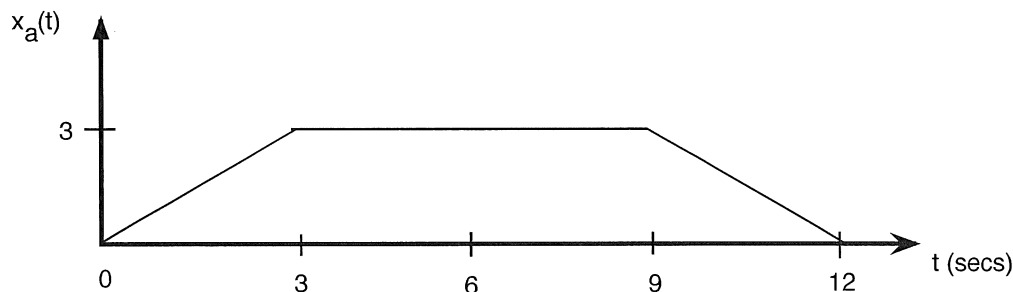
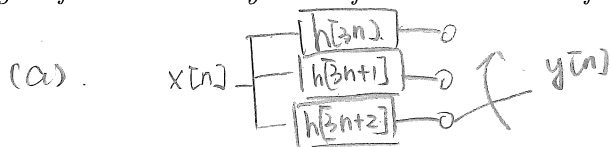


Figure 2.

- (i) For the ideal case where $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y[n]$ of the system in Figure 1, when $x[n]$ is input to the system. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

- (ii) For the ideal case where $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_0[n] = x[n]*h_0[n]$, when $x[n]$ is input to the filter $h_0[n] = h[3n]$. Write output in sequence form (indicating where is $n = 0$ OR do stem plot).
- (iii) For the ideal case where $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_1[n] = x[n]*h_1[n]$, when $x[n]$ is input to the filter $h_1[n] = h[3n + 1]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.
- (iv) For the ideal case where $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_2[n] = x[n]*h_2[n]$, when $x[n]$ is input to the filter $h_2[n] = h[3n + 2]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

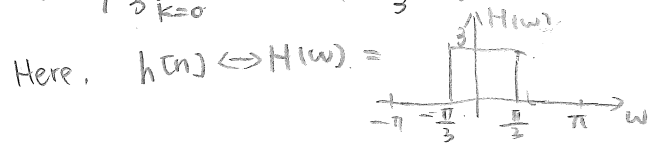
This page left intentionally blank for student work for Problem 1.



(b) (i) $h[3n] \leftrightarrow H_0(\omega) = \left\{ \frac{1}{L} \sum_{k=0}^{L-1} e^{-j\frac{k\omega}{L}} H\left(\frac{\omega-2\pi k}{L}\right) \right\} e^{j\frac{k}{L}\omega}$

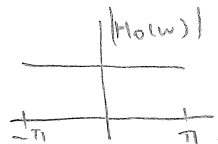
$L=3, \ell=0$

$\therefore H_0(\omega) = \left\{ \frac{1}{3} \sum_{k=0}^2 e^{j0} H\left(\frac{\omega-2\pi k}{3}\right) \right\} e^{j\frac{0}{3}\omega}$



only $k=0$ contribute to $H_0(\omega)$ in $-\pi < \omega < \pi$

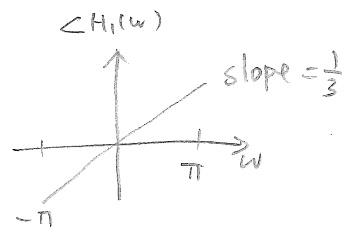
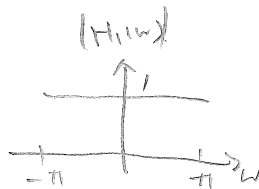
So $H_0(\omega) = 1$ in $-\pi < \omega < \pi$



(ii) $h_1[n] = h[3n+1] \xrightarrow{\text{DTFT}} H_1(\omega) = \left\{ \frac{1}{3} \sum_{k=0}^2 e^{-j\frac{k\omega}{3} \cdot 1} H\left(\frac{\omega-2\pi k}{3}\right) \right\} e^{j\frac{1}{3}\omega}$

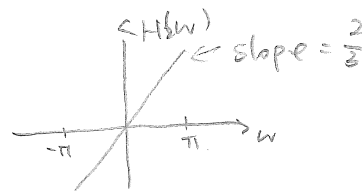
(iii) only $k=0$ contribute to $-\pi < \omega < \pi$ for $H_1(\omega)$

$H_1(\omega) = e^{j\frac{1}{3}\omega}$



(iv) $H_2(\omega) = \left\{ \frac{1}{3} \sum_{k=0}^2 e^{-j\frac{k\omega}{3} \cdot 2} H\left(\frac{\omega-2\pi k}{3}\right) \right\} e^{j\frac{2}{3}\omega}$

(v) only $k=0$ contribute to $-\pi < \omega < \pi$ for $H_2(\omega)$ $H_2(\omega) = e^{j\frac{2}{3}\omega}$



This page left intentionally blank for student work for Problem 1.

(c) (i) the system upsample $x[n]$ by 3.

$$y[n] = X_a\left[\frac{nT_s}{3}\right] = X_a\left[\frac{n \cdot 3}{3}\right] = X_a[n]$$

$$\therefore y[n] = \left\{ \underset{\substack{\uparrow \\ n=0}}{0} \quad 1 \quad 2 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 2 \quad 1 \quad 0 \right\}$$

(ii) $y[3n]$ is the output of $h[3n]$

$$y[3n] = \{ 0 \quad 3 \quad 3 \quad 3 \quad 0 \}$$

(iii) $y[3n+1]$ is the output of $h[3n+1]$

$$y[3n+1] = \{ 1 \quad 3 \quad 3 \quad 2 \quad 0 \}$$

(iv) $y[3n+2]$ is the output of $h[3n+2]$

$$y[3n+2] = \{ 2 \quad 3 \quad 3 \quad 1 \quad 0 \}$$

Problem 2.

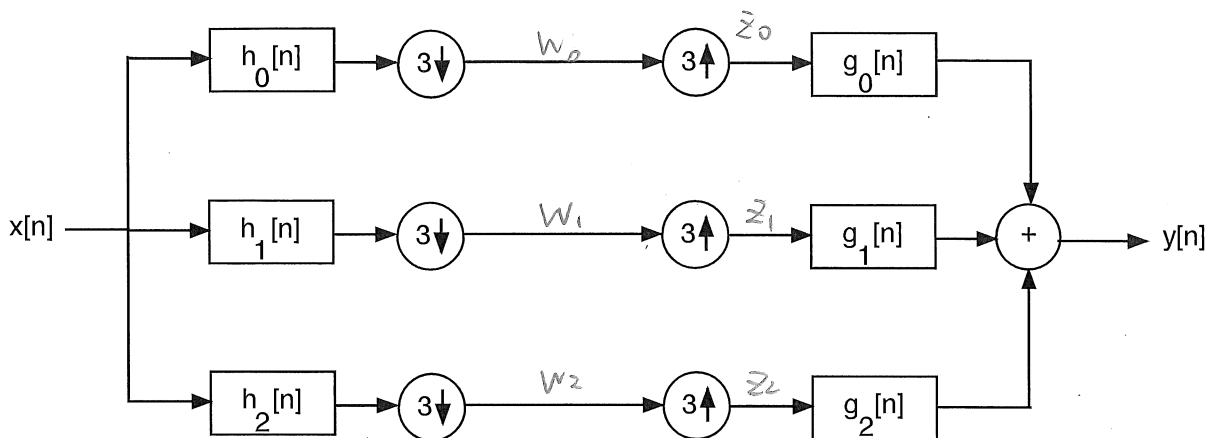


Figure 3.

In the frequency domain, the DTFT of the output is related to the DTFT at the input according to the expression below.

$$Y(\omega) = A(\omega)X(\omega) + B(\omega)X\left(\omega - \frac{2\pi}{3}\right) + C(\omega)X\left(\omega - \frac{4\pi}{3}\right)$$

- (a) Determine expressions for $A(\omega)$, $B(\omega)$, and $C(\omega)$ in terms of $H_0(\omega)$, $H_1(\omega)$, and $H_2(\omega)$ and $G_0(\omega)$, $G_1(\omega)$, and $G_2(\omega)$.
- (b) Consider the case where $g_k[n] = h_k^*[-n]$, $k = 0, 1, 2$. Determine expressions for $A(\omega)$, $B(\omega)$, and $C(\omega)$ in terms of $H_0(\omega)$, $H_1(\omega)$, and $H_2(\omega)$.
- (c) Consider the case where $g_k[n] = h_k^*[-n]$, $k = 0, 1, 2$ AND $h_k[n] = e^{jk\frac{2\pi}{3}n}h[n]$, $k = 0, 1, 2$. Determine expressions for $A(\omega)$, $B(\omega)$, and $C(\omega)$ all in terms of $H(\omega)$.
- (d) Consider the case where $g_k[n] = h_k^*[-n]$, $k = 0, 1, 2$ AND $h_k[n] = e^{jk\frac{2\pi}{3}n}h[n]$, $k = 0, 1, 2$ AND $h[n] = u[n] - u[n - 3]$. Show that $A(\omega) = 9$, $B(\omega) = 0$, and $C(\omega) = 0$ for all ω .
- (e) Under the conditions stated in part (d) directly above, develop and draw the efficient implementation of the entire Filter Bank in Figure 3. Show as much detail as possible. Using the efficient implementation, explain how the system is a Perfect Reconstruction Filter Bank.

This page left intentionally blank for student work for Problem 2.

$$\begin{aligned}
 (a) \quad W_i(w) &= \frac{1}{3} \sum_{k=0}^2 H_i\left(\frac{w-k2\pi}{3}\right) X\left(\frac{w-2\pi k}{3}\right) \\
 z_i(w) &= W_i(3w) = \frac{1}{3} \sum_{k=0}^2 H_i\left(w - \frac{2\pi}{3}k\right) X\left(w - \frac{2\pi}{3}k\right) \\
 Y(w) &= \sum_{i=0}^2 Y_i(w) \\
 &= \frac{1}{3} \sum_{i=0}^2 \sum_{k=0}^2 G_i(w) H_i\left(w - \frac{k2\pi}{3}\right) X\left(w - \frac{k2\pi}{3}\right) \\
 &= \frac{1}{3} \left\{ G_0(w)H_0(w) + G_1(w)H_1(w) + G_2(w)H_2(w) \right\} X(w) \quad \leftarrow A(w) \\
 &\quad + \frac{1}{3} \left\{ G_0(w)H_0\left(w - \frac{2\pi}{3}\right) + G_1(w)H_1\left(w - \frac{2\pi}{3}\right) + G_2(w)H_2\left(w - \frac{2\pi}{3}\right) \right\} X\left(w - \frac{2\pi}{3}\right) \quad \leftarrow B(w) \\
 &\quad + \frac{1}{3} \left\{ G_0(w)H_0\left(w - \frac{4\pi}{3}\right) + G_1(w)H_1\left(w - \frac{4\pi}{3}\right) + G_2(w)H_2\left(w - \frac{4\pi}{3}\right) \right\} X\left(w - \frac{4\pi}{3}\right) \quad \leftarrow C(w)
 \end{aligned}$$

(b) $G_k(w) = H_k^*(w)$ because $g_k[n] = h_k^*[L-n]$

$$A(w) = \frac{1}{3} \sum_{k=0}^2 G_k(w) H_k(w) = \frac{1}{3} \sum_{k=0}^2 H_k^*(w) H_k(w) = \frac{1}{3} \sum_{k=0}^2 |H_k(w)|^2$$

$$B(w) = \frac{1}{3} \sum_{k=0}^2 G_k(w) H_k\left(w - \frac{2\pi}{3}\right) = \frac{1}{3} \sum_{k=0}^2 H_k^*(w) H_k\left(w - \frac{2\pi}{3}\right)$$

$$C(w) = \frac{1}{3} \sum_{k=0}^2 G_k(w) H_k\left(w - \frac{4\pi}{3}\right) = \frac{1}{3} \sum_{k=0}^2 H_k^*(w) H_k\left(w - \frac{4\pi}{3}\right)$$

(c) $B(w) = \frac{1}{3} \left\{ H^*(w) H\left(w - \frac{2\pi}{3}\right) + H^*\left(w - \frac{2\pi}{3}\right) H\left(w - \frac{4\pi}{3}\right) + H^*\left(w - \frac{4\pi}{3}\right) H\left(w - 2\pi\right) \right\}$

because $H_k(w) = H\left(w - k\frac{2\pi}{3}\right)$

$H(w)$ is periodic with period 2π .

so $B(w) = \frac{1}{3} \left\{ H^*(w) H\left(w - \frac{2\pi}{3}\right) + H^*\left(w - \frac{2\pi}{3}\right) H\left(w - \frac{4\pi}{3}\right) + H^*\left(w - \frac{4\pi}{3}\right) H(w) \right\}$

$$C(w) = \frac{1}{3} \left\{ H^*(w) H\left(w - \frac{4\pi}{3}\right) + H^*\left(w - \frac{2\pi}{3}\right) H\left(w - 2\pi\right) + H^*\left(w - \frac{4\pi}{3}\right) H\left(w - \frac{8\pi}{3}\right) \right\}$$

$$H\left(w - \frac{8\pi}{3}\right) = H\left(w - \frac{2\pi}{3}\right)$$

$$\therefore C(w) = \frac{1}{3} \left\{ H^*(w) H\left(w - \frac{4\pi}{3}\right) + H^*\left(w - \frac{2\pi}{3}\right) H\left(w - 2\pi\right) + H^*\left(w - \frac{4\pi}{3}\right) H\left(w - \frac{2\pi}{3}\right) \right\}$$

$$A(w) = \frac{1}{3} \left\{ H^*(w) H(w) + H^*\left(w - \frac{2\pi}{3}\right) H\left(w - \frac{2\pi}{3}\right) + H^*\left(w - \frac{4\pi}{3}\right) H\left(w - \frac{4\pi}{3}\right) \right\}$$

This page left intentionally blank for student work for Problem 2.

(cd) $h[n] = u[n] - u[n-3]$ $H(\omega) = \frac{\sin \frac{3}{2}\omega}{\sin \frac{1}{2}\omega} e^{-j\frac{1}{2}\omega}$ $|H(\omega)| = 3$ $h[n] \xrightarrow{\text{DFT}} 3\delta[k]$
 $\uparrow H_3(k)$

$$A(\omega) = \frac{1}{3} \left\{ H^*(\omega) H(\omega) + H^*(\omega - \frac{2\pi}{3}) H(\omega - \frac{2\pi}{3}) + H^*(\omega - \frac{4\pi}{3}) H(\omega - \frac{4\pi}{3}) \right\}$$

$$= \frac{1}{3} \left\{ |H(\omega)|^2 + |H(\omega - \frac{2\pi}{3})|^2 + |H(\omega - \frac{4\pi}{3})|^2 \right\}$$

(If computed as 3-pt DFT: $A_3(k) = \frac{1}{3} \{ 3\delta[k] + 3\delta[k-1] + 3\delta[k-2] \} = 3$)

$$= \frac{1}{3} \{ 9 + 9 + 9 \} = 9$$

$$B(\omega) = \frac{1}{3} \left\{ H^*(\omega) H(\omega - \frac{2\pi}{3}) + H^*(\omega - \frac{2\pi}{3}) H(\omega - \frac{4\pi}{3}) + H^*(\omega - \frac{4\pi}{3}) H(\omega) \right\}$$

$$= \frac{1}{3} \left\{ \frac{\sin \frac{3}{2}\omega}{\sin \frac{1}{2}\omega} e^{j\omega} \frac{\sin \frac{3}{2}(\omega - \frac{2\pi}{3})}{\sin \frac{1}{2}(\omega - \frac{2\pi}{3})} e^{-j(\omega - \frac{2\pi}{3})} + \frac{\sin \frac{3}{2}(\omega - \frac{2\pi}{3})}{\sin \frac{1}{2}(\omega - \frac{2\pi}{3})} e^{j(\omega - \frac{2\pi}{3})} \frac{\sin \frac{3}{2}(\omega - \frac{4\pi}{3})}{\sin \frac{1}{2}(\omega - \frac{4\pi}{3})} e^{-j(\omega - \frac{4\pi}{3})} \right.$$

(If computed as 3-pt DFT:)

$$B_3(k) = \frac{1}{3} \{ 3\delta[k] \cdot 3\delta[k-1] + 3\delta[k-1] \cdot 3\delta[k-2] + 3\delta[k-2] \cdot 3\delta[k] \}$$

$$= 0$$

$$C_3(k) = \frac{1}{3} \{ 3\delta[k] \cdot 3\delta[k-2] + 3\delta[k-1] \cdot 3\delta[k] + 3\delta[k-2] \cdot 3\delta[k-1] \}$$

$$= 0$$

$$+ \frac{\sin \frac{3}{2}(\omega - \frac{4\pi}{3})}{\sin \frac{1}{2}(\omega - \frac{4\pi}{3})} e^{j(\omega - \frac{4\pi}{3})} \frac{\sin \frac{3}{2}\omega}{\sin \frac{1}{2}\omega} e^{-j\omega} \}$$

$$= \frac{1}{3} \left\{ \frac{\sin \frac{3}{2}\omega}{\sin \frac{1}{2}\omega} \frac{\sin \frac{3}{2}(\omega - \frac{2\pi}{3})}{\sin \frac{1}{2}(\omega - \frac{2\pi}{3})} e^{j\frac{2\pi}{3}} + \frac{\sin \frac{3}{2}(\omega - \frac{2\pi}{3})}{\sin \frac{1}{2}(\omega - \frac{2\pi}{3})} \frac{(-\sin \frac{3}{2}\omega)}{\sin \frac{1}{2}(\omega - \frac{4\pi}{3})} e^{j\frac{2\pi}{3}} \right.$$

$$\left. + \frac{(-\sin \frac{3}{2}\omega)}{\sin \frac{1}{2}(\omega - \frac{4\pi}{3})} \frac{\sin \frac{3}{2}\omega}{\sin \frac{1}{2}\omega} e^{j\frac{2\pi}{3}} \right\}$$

$$\left(\frac{\sin \frac{3}{2}(\omega - \frac{2\pi}{3})}{\sin \frac{1}{2}(\omega - \frac{2\pi}{3})} = \sin(\frac{3}{2}\omega) \cos \pi - \cos \frac{3}{2}\omega \sin \pi = \sin(\frac{3}{2}\omega) \cos \pi = \sin(\frac{3}{2}\omega) \cdot (-1) = -\sin(\frac{3}{2}\omega) \right)$$

$$\sin \frac{3}{2}(\omega - \frac{4\pi}{3}) = \sin \frac{3}{2}\omega \cos 2\pi - \cos \frac{3}{2}\omega \sin 2\pi = \sin(\frac{3}{2}\omega)$$

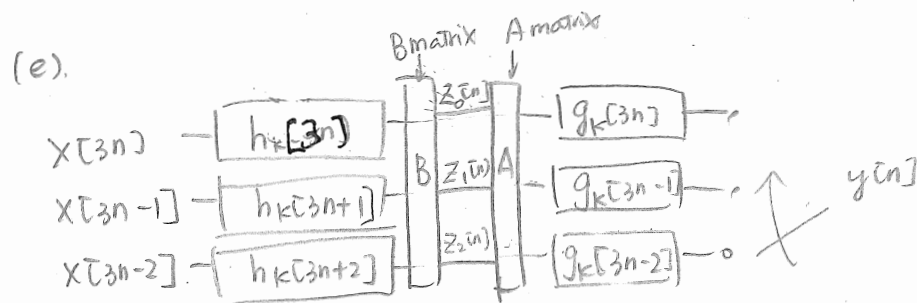
$$= \frac{1}{3} \cdot e^{j\frac{2\pi}{3}} \left\{ \frac{(\sin(\frac{3}{2}\omega))^2}{\sin(\frac{1}{2}\omega) \sin \frac{1}{2}(\omega - \frac{2\pi}{3})} + \frac{-(\sin(\frac{3}{2}\omega))^2}{\sin \frac{1}{2}(\omega - \frac{2\pi}{3}) \sin \frac{1}{2}(\omega - \frac{4\pi}{3})} + \frac{-(\sin(\frac{3}{2}\omega))^2}{\sin \frac{1}{2}(\omega - \frac{4\pi}{3}) \sin \frac{1}{2}\omega} \right\}$$

the aliasings cancel each other.
 numerator of B(ω) → $\frac{1}{8} \sin^2 \omega + \frac{\sqrt{3}}{8} \sin^2 \omega \cos \omega - \frac{\sqrt{3}}{8} \cos^2 \omega \sin \omega + \frac{1}{8} \sin^2 \omega + \frac{\sqrt{3}}{8} \sin^2 \omega \cos \omega - \frac{\sqrt{3}}{8} \cos^2 \omega \sin \omega + \frac{1}{8} \sin^2 \omega + \frac{\sqrt{3}}{8} \sin^2 \omega \cos \omega - \frac{\sqrt{3}}{8} \cos^2 \omega \sin \omega = 0$

$$C(\omega) = \frac{1}{3} \left\{ H^*(\omega) H(\omega - \frac{4\pi}{3}) + H^*(\omega - \frac{4\pi}{3}) H(\omega) + H^*(\omega - \frac{4\pi}{3}) H(\omega - \frac{2\pi}{3}) \right\}$$

the aliasings cancel each other.
 $\therefore C(\omega) = 0$
 $H(\omega - \frac{4\pi}{3}) = H(\omega + \frac{2\pi}{3})$
 each term has a phase difference of $\frac{2\pi}{3}$
 $H(\omega)$ and $H(\omega + \frac{2\pi}{3})$
 $H(\omega - \frac{2\pi}{3})$ and $H(\omega)$
 $H(\omega - \frac{4\pi}{3})$ and $H(\omega - \frac{2\pi}{3})$

This page left intentionally blank for student work for Problem 2.



Since $h[n]$ is of length 3.

so each $h_k[n] = \delta[n]$

$h_k[n+1] = \delta[n]$

$h_k[n+2] = \delta[n]$

$g_k[n] = h_k^*[n]$

$g_k[n-1] = h_k^*[-(n-1)] = h_k^*[1-n]$

$g_k[n] = \delta[n]$

$g_k[n-1] = \delta[n]$

$g_k[n-2] = \delta[n]$

$$h_k[n+l] = e^{jk\frac{2\pi}{3}(3n+l)} h[n]$$

$$= e^{jk\frac{2\pi}{3} \cdot 3n} e^{jk\frac{2\pi}{3} \cdot l} h[n]$$

$$\begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{3}} & e^{j\frac{4\pi}{3}} \\ 1 & e^{j\frac{4\pi}{3}} & e^{j\frac{2\pi}{3}} \end{bmatrix} \begin{bmatrix} x[3n] \\ x[3n-1] \\ x[3n-2] \end{bmatrix}$$

matrix B

$$g_k[n-1] = h_k^*[-(n-1)] = e^{jk\frac{2\pi}{3}(1-3n)} h[n] = e^{jk\frac{2\pi}{3} \cdot 3n} e^{jk\frac{2\pi}{3} \cdot 1} h[n]$$

$$\begin{bmatrix} y[3n] \\ y[3n-1] \\ y[3n-2] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{3}} & e^{j\frac{4\pi}{3}} \\ 1 & e^{j\frac{4\pi}{3}} & e^{j\frac{2\pi}{3}} \end{bmatrix} \begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \end{bmatrix}$$

matrix A

Problem 3. Let $x[n]$ be of length $L = 7$, i.e., $x[n] = 0$ for $n < 0$ and $n \geq 7$, and $h[n]$ also be of length $M = 7$. Let $X_8(k)$ and $H_8(k)$ denote 8-point DFT's of $x[n]$ and $h[n]$, respectively. The 8-point inverse DFT of the product $Y_8(k) = X_8(k)H_8(k)$, denoted $y_8[n]$, produces the following values:

n	0	1	2	3	4	5	6	7
$y_8[n]$	-2	0	-2	0	-2	0	7	0

Let $X_{10}(k)$ and $H_{10}(k)$ denote the 10-point DFT's of the aforementioned sequences $x[n]$ and $h[n]$. The 10-point inverse DFT of the product $Y_{10}(k) = X_{10}(k)H_{10}(k)$, denoted $y_{10}[n]$, produces the following values:

n	0	1	2	3	4	5	6	7	8	9
$y_{10}[n]$	-2	0	-2	0	-1	0	7	0	-1	0

Given $y_8[n]$ and $y_{10}[n]$, find the **linear** convolution of $x[n]$ and $h[n]$, i. e., list all of the numerical values of $y[n] = x[n] * h[n]$. **You must briefly explain and show all work.**

$7+7-1=13 > 8 \rightarrow 13-8=5$ point aliasing
 $13 > 10 \rightarrow 13-10=3$ point aliasing.

$y_{10}[n]$ total 13 points

assuming $y[n] = x[n] * h[n] = \{ y[0] \ y[1] \ y[2] \ 0 \ -1 \ 0 \}$
 $\{ y[11] \ y[12] \}$

because we know in 10-pt DFT, the first 3 is aliased, the next 7 isn't aliased.

in 8pt DFT, 5 point in the front is aliased into 5 point at end

So

$$\begin{aligned} & \{ y[0] \ y[1] \ y[2] \ 0 \ -1 \} \\ & + \{ -1 \ 0 \ y[10] \ y[11] \ y[12] \} \\ & = \{ (y[0]-1) \ (y[1]) \ (y[2]+y[10]) \ y[11] \ (y[12]-1) \} = \{ -2 \ 0 \ -2 \ 0 \ -2 \} \end{aligned}$$

So $y[0]-1 = -2 \rightarrow y[0] = -1$

$y[1] = 0$

$y[2]+y[10] = -2$

$y[11] = 0$

$y[12]-1 = -2$

$y[12] = -1$

This page left intentionally blank for student work for Problem 3.

$$\text{So, } y[n] = \{ -1, 0, y[2], 0, -1, 0, 1, 0, -1, 0, y[10], 0, -1 \}$$

in 10-pt DFT, 3 point at the beginning
is aliased into 3 point at end

$$\begin{aligned} & \{ -1, 0, y[2] \} \\ + & \{ y[10], 0, -1 \} = \{ (y[10]-1), 0, (y[2]-1) \} \\ & = \{ -2, 0, -2 \} \end{aligned}$$

$$\therefore y[10]-1 = -2$$

$$y[10] = -1$$

$$y[2]-1 = -2$$

$$y[2] = -1$$

$$\therefore y[n] = \{ -1, 0, -1, 0, -1, 0, 1, 0, -1, 0, -1, 0, -1 \}$$

Problem 4.

- (a) Let $X_8(k) = X(2\pi k/8)$, where $X(\omega)$ is the DTFT of the sequence

$$x[n] = (0.9)^n u[n] \xleftrightarrow{DTFT} X(\omega) = \frac{1}{1 - 0.9e^{-j\omega}}$$

That is, $X_8(k)$ is what we obtain by sampling $X(\omega)$ at $N = 8$ equi-spaced points in the interval $0 \leq \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 8-pt inverse DFT of $X_8(k)$ may be expressed as

$$x_8[n] = \sum_{\ell=-\infty}^{\infty} x[n - \ell 8] \{u[n] - u[n - 8]\} \xleftrightarrow{DFT} X_8(k) = \frac{1}{1 - 0.9e^{-j2\pi k/8}}; k = 0, 1, \dots, 7$$

Determine a simple, closed-form expression for $x_8[n]$. A *closed-form* expression contains NO summations and it is NOT a listing of numbers. *Hint:*

$$\frac{1}{1 - (.9)^8} = 1.7558$$

- (b) Consider normalizing $x_8[n]$ so that it's first value is one, $\tilde{x}_8[n] = x_8[n]/x_8[0]$, $n = 0, 1, \dots, 7$. Compare $\tilde{x}_8[n]$ and $x[n]$ over $n = 0, 1, \dots, 7$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.
- (c) Let $Y_8(k) = Y(2\pi k/8)$, where $Y(\omega)$ is the DTFT of the sequence

$$y[n] = (0.9)^n u[n] + (0.8)^n u[n]$$

That is, $Y_8(k)$ is what we obtain by sampling $Y(\omega)$ at $N = 8$ equi-spaced points in the interval $0 \leq \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 8-pt inverse DFT of $Y_8(k)$ may be expressed as

$$y_8[n] = \sum_{\ell=-\infty}^{\infty} y[n - \ell 8] \{u[n] - u[n - 8]\} \xleftrightarrow{DFT} Y_8(k) = \frac{1}{1 - 0.9e^{-j2\pi k/8}} + \frac{1}{1 - 0.8e^{-j2\pi k/8}}$$

Determine a simple, closed-form expression for $y_8[n]$. A *closed-form* expression contains NO summations and it is NOT a listing of numbers. *Hint:*

$$\frac{1}{1 - (.8)^8} = 1.2016$$

- (d) Consider normalizing $y_8[n]$ so that it's first value is one, $\tilde{y}_8[n] = y_8[n]/y_8[0]$, $n = 0, 1, \dots, 7$. Compare $\tilde{y}_8[n]$ and $y[n]$ over $n = 0, 1, \dots, 7$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.

This page left intentionally blank for student work for Problem 4.

(a)

$$\begin{aligned}
 x_8[n] &= \sum_{l=-\infty}^0 x[n-8l] \{u[n] - u[n-8]\} \\
 &= \sum_{l=0}^{\infty} x[n+8l] \{u[n] - u[n-8]\} \\
 &= \sum_{l=0}^{\infty} (0.9)^{n+8l} \{u[n] - u[n-8]\} \\
 &= (0.9)^n \sum_{l=0}^{\infty} (0.9)^{8l} \{u[n] - u[n-8]\} \\
 &= (0.9)^n \frac{1}{1-(0.9)^8} \{u[n] - u[n-8]\} \\
 &= 1.7558 (0.9)^n \{u[n] - u[n-8]\}
 \end{aligned}$$

(b)

$$x_8[0] = \frac{1}{1-(0.9)^8} = 1.7558$$

$$x_8[n] = x_8[n] / x_8[0] = (0.9)^n \{u[n] - u[n-8]\}$$

$x_8[n]$ and $x[n]$ are the same for $n=0, 1, \dots, 7$.

(c)

$$\begin{aligned}
 y_8[n] &= \sum_{l=-\infty}^{\infty} y[n-8l] \{u[n] - u[n-8]\} \\
 &= \sum_{l=-\infty}^0 (0.9)^{n+8l} + (0.8)^{n+8l} \{u[n] - u[n-8]\} \\
 &= \sum_{l=0}^{\infty} (0.9)^{n+8l} + (0.8)^{n+8l} \{u[n] - u[n-8]\} \\
 &= \left[(0.9)^n \sum_{l=0}^{\infty} (0.9)^{8l} + (0.8)^n \sum_{l=0}^{\infty} (0.8)^{8l} \right] \{u[n] - u[n-8]\} \\
 &= \left[(0.9)^n / (1-(0.9)^8) + \frac{(0.8)^n}{1-(0.8)^8} \right] \{u[n] - u[n-8]\} \\
 &= \left(1.7558 (0.9)^n + 1.2016 (0.8)^n \right) \{u[n] - u[n-8]\}.
 \end{aligned}$$

This page left intentionally blank for student work for Problem 4.

d)

$$y_8[0] = 1.7558 + 1.2016$$

$$\tilde{y}_8[n] = y_8[n] / y_8[0] = \frac{\{1.7558(0.9)^n + 1.2016(0.8)^n\}}{1.7558 + 1.2016} \quad \{u[n] - u[n-8]\}$$

$$= \left\{ \left(\frac{1.7558}{1.7558 + 1.2016} \right) (0.9)^n + \left(\frac{1.2016}{1.7558 + 1.2016} \right) (0.8)^n \right\} \{u[n] - u[n-8]\}$$

$$\neq y_8[n] \quad n = 0, 1, \dots, 7.$$