NAME:

11 December 2013

Digital Signal Processing I **Final Exam** **Fall 2013**

Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes. Three 8.5 x 11 crib sheets allowed
Calculators NOT allowed.
This test contains **four** problems.
All work should be done on the blank pages provided.
Your answer to each part of the exam should be clearly labeled.

Problem 1. [50 points] System 1 and System 2 defined in parts (a) and (b), respectively, will eventually be connected in parallel. We first analyze them individually.

(a) Consider System 1 below:

$$\text{System 1:} \quad y_1[n] = 0.9j\, y_1[n-1] - 0.9j\, x[n] - x[n-1]$$

(i) Determine the Transfer Function for System 1, denoted $H_1(z)$. $H_1(z)$ is the Z-Transform of the impulse response, $h_1[n]$, for System 1, although you can find $H_1(z)$ anyway you like. Identify the poles and zeros, and do a pole-zero plot.

(ii) Determine the frequency response of System 1, denoted $H_1(\omega)$. $H_1(\omega)$ is the DTFT of the impulse response, $h_1[n]$, for System 1, although you can find $H_1(\omega)$ anyway you like. Plot the magnitude $|H_1(\omega)|$ over $-\pi < \omega < \pi$.

(iii) Determine the autocorrelation of the impulse response $h_1[n]$. Do a stem plot of $r_{h_1h_1}[\ell]$.

(b) Consider System 2 below:

$$\text{System 2:} \quad y_2[n] = -0.9j\, y_2[n-1] - 0.9j\, x[n] + x[n-1]$$

(i) Determine the Transfer Function for System 2, denoted $H_2(z)$. $H_2(z)$ is the Z-Transform of the impulse response, $h_2[n]$, for System 2, although you can find $H_2(z)$ anyway you like. Identify the poles and zeros, and do a pole-zero plot.

(ii) Determine the frequency response of System 2, denoted $H_2(\omega)$. $H_2(\omega)$ is the DTFT of the impulse response, $h_2[n]$, for System 2, although you can find $H_2(\omega)$ anyway you like. Plot the magnitude $|H_2(\omega)|$ over $-\pi < \omega < \pi$.

(iii) Determine the autocorrelation of the impulse response $h_2[n]$. Do a stem plot of $r_{h_2h_2}[\ell]$.

(c) The overall system is formed from connecting System 1 and System 2 in parallel.

(i) Determine the Transfer Function for the overall system, denoted $H(z)$. $H(z)$ is the Z-Transform of the impulse response, $h[n]$, for the parallel combination of Systems 1 and 2. You can find $H(z)$ anyway you like. Identify the poles and zeros for the overall system, and do a pole-zero plot.

(ii) Determine the frequency response of the overall system, denoted $H(\omega)$. $H(\omega)$ is the DTFT of the impulse response, $h[n]$, for the overall system; you can find $H(\omega)$ anyway you like. Plot the magnitude $|H(\omega)|$ over $-\pi < \omega < \pi$. showing as much detail as possible. Point out any frequencies for which $H(\omega) = 0$.

(iii) Determine the output of the overall system, $y[n]$, when the input is the DT signal below. The overall system is the parallel combination of Systems 1 and 2.

$$x[n] = 1 + 3\cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right) + 3(-1)^n \qquad -\infty < n < \infty$$

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Problem 2.
Continuous-Time Fourier Transform (rads/sec): $X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Continuous-Time Fourier Transform Pair (rads/sec): $\mathcal{F}\left\{\frac{\sin(Wt)}{\pi t}\right\} = rect\left\{\frac{\omega}{2W}\right\}$

where $rect(x) = 1$ for $|x| < 0.5$ and $rect(x) = 0$ for $|x| > 0.5$.
Continuous-Time Fourier Transform Property: $\mathcal{F}\{x_1(t)x_2(t)\} = \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$,
where $*$ denotes convolution, and $\mathcal{F}\{x_i(t)\} = X_i(\omega)$, $i = 1, 2$.
Relationship between DTFT & CTFT frequency variables in rads/sec: $\omega = \Omega T_s$
Relationship between DTFT and CTFT frequency variables in Hz: $\omega = 2\pi\frac{F}{F_s}$,
where $F_s = \frac{1}{T_s}$ is the sampling rate in Hz

Problem 1 (a). The impulse response of an analog filter which passes up to the maximum frequency $\omega_M = 20$ rads/sec is

$$g_a(t) = \frac{2\pi}{40}\frac{\sin\left(20(t - \frac{\pi}{40})\right)}{\pi(t - \frac{\pi}{40})}$$

This impulse response is sampled at the Nyquist rate $\omega_s = 40$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{40}$ sec, to form the discrete-time filter

$$g[n] = g_a(nT_s) \quad \text{where:} \quad T_s = \frac{2\pi}{40}$$

(a) Determine $G(\omega)$, the DTFT of $g[n]$. Plot both the magnitude $|G(\omega)|$ and the phase $\angle G(\omega)$ (separate plots) over $-\pi < \omega < \pi$.

(b) Consider the continuous-time signal $x_a(t)$ below. A discrete-time signal is created by sampling $x(t)$ according to $x[n] = x_a(nT_s)$ for $T_s = \frac{2\pi}{40}$. Plot the magnitude of the DTFT of $x[n]$, $|X(\omega)|$, over $-\pi < \omega < \pi$.

$$x_a(t) = T_s\frac{\pi}{4}\left\{\frac{\sin(4t)}{\pi t}\,\frac{\sin(16t)}{\pi t}\right\}$$

(c) $x[n]$ is passed through the filter $g[n]$ to form the output

$$y[n] = x[n] * g[n]$$

A reconstructed signal is formed from $y[n]$ above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $y_r(t)$.

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]h(t - nT_s) \quad \text{where:} \quad T_s = \frac{2\pi}{40} \quad \text{and} \quad h(t) = \frac{\sin(20t)}{\pi t}$$

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$$g[n] = T_s \frac{\sin\left[20\left(nT_s - \frac{T_s}{2}\right)\right]}{\pi\left(nT_s - \frac{T_s}{2}\right)} \qquad \text{since } T_s = \frac{2\pi}{40}$$

$$= \frac{2\pi}{40} \frac{\sin\left[20\left(n-\frac{1}{2}\right)\frac{2\pi}{40}\right]}{\pi\left(n-\frac{1}{2}\right)\frac{2\pi}{40}} = \frac{\sin\left[\pi\left(n-\frac{1}{2}\right)\right]}{\pi\left(n-\frac{1}{2}\right)}$$

$$= \frac{\sin\left[\frac{\pi}{2}n\right]}{\frac{\pi}{2}n}\Bigg|_{\underbrace{n = 2n-1}_{\text{polyphase component}}}$$

We derived formula in class

$h_\ell^-[n] = [Ln-\ell)$, then

$$H_\ell^-(\omega) = \left\{\frac{1}{L}\sum_{k=0}^{L-1} e^{+j\frac{k2\pi\ell}{L}} H\left(\frac{\omega - k2\pi}{L}\right)\right\} e^{-j\frac{\ell}{L}\omega}$$

For the ideal case ~~at~~ hand $L=2$ and only $k=0$ term contributes in $-\pi < \omega < \pi$

$$G(\omega) = e^{-j\frac{\omega}{2}} \quad \text{for } -\pi < \omega < \pi$$

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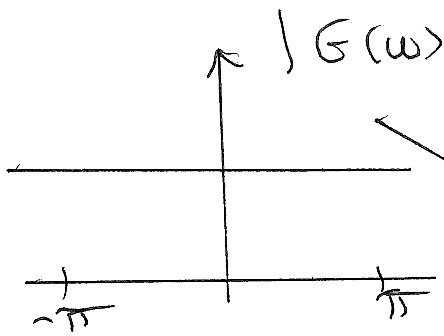
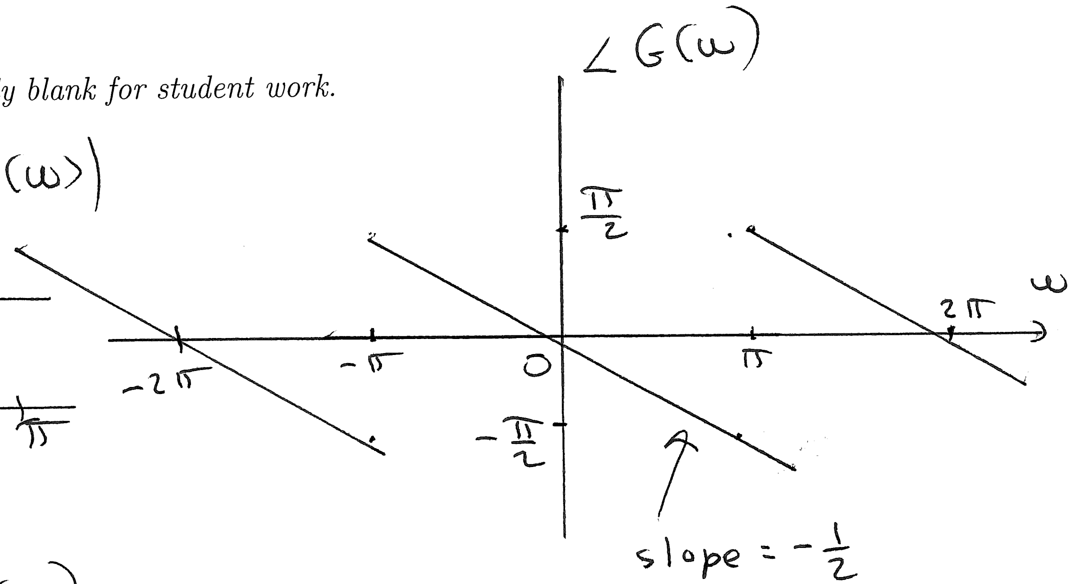
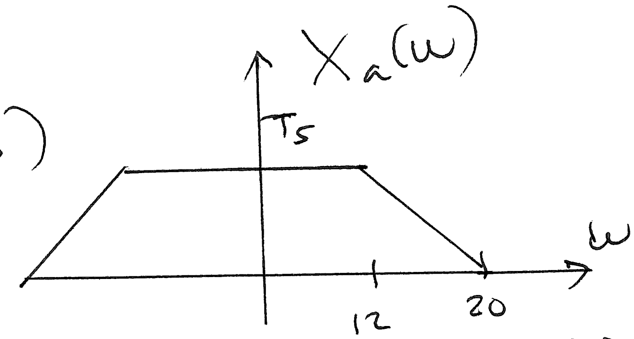




(b)

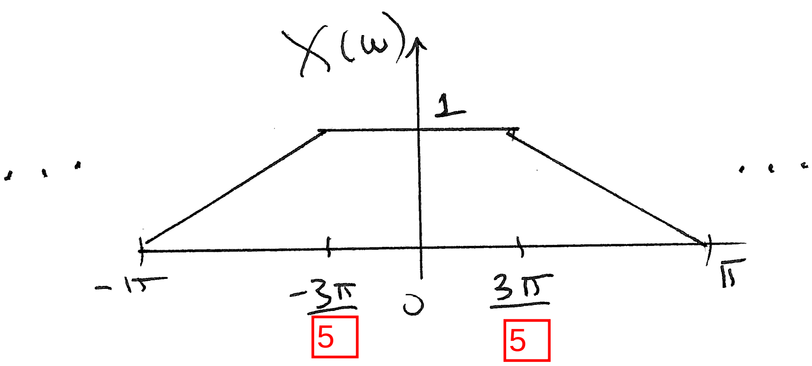


$\omega_s = 40$ in rads/sec

$\Rightarrow \quad 20T_s = 20\,\frac{2\pi}{40} = \pi$

$12\,T_s = 12\,\frac{2\pi}{40}$

$= \frac{3\pi}{5}$



(c) Filter g[n] effects a fractional delay of $\frac{T_s}{2}$

answer: $y_r(t) = x_a\left(t - \frac{T_s}{2}\right)$

since it's perfect reconstruction

Nyquist rate sampling and Ideal LPF

passing −20 to 20 perfectly

Problem 3. Consider a causal FIR filter of length $M = 9$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \frac{\sin\left[\pi\left(n+\frac{1}{2}+\ell 9\right)\right]}{\pi\left(n+\frac{1}{2}+\ell 9\right)} \{u[n]-u[n-9]\}$$

(a) Determine the 9-pt DFT of $h_p[n]$, denoted $H_9(k)$, for $0 \le k \le 8$. You can EITHER write an expression for $H_9(k)$, OR list the numerical values: $H_9(0) =?$, $H_9(1) =?$, $H_9(2) =?$, $H_9(3) =?$, $H_9(4) =?$, $H_9(5) =?$, $H_9(6) =?$, $H_9(7) =?$, $H_9(8) =?$.

(b) Consider the sequence $x[n]$ of length $L = 9$ below, equal to a sum of 9 finite-length sinewaves.

$$x[n] = \sum_{k=0}^{8} e^{jk\frac{2\pi}{9}n} \{u[n]-u[n-9]\}$$

$y_9[n]$ is formed by computing $X_9(k)$ as an 9-pt DFT of $x[n]$, $H_9(k)$ as a 9-pt DFT of $h[n]$ and, finally, then $y_9[n]$ is computed as the 9-pt inverse DFT of $Y_9(k) = X_9(k)H_9(k)$. Express the result $y_9[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

Next, consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} 16j \frac{\sin\left[\frac{3\pi}{8}(n+\ell 8)\right]}{\pi(n+\ell 8)} \frac{\sin\left[\frac{\pi}{8}(n+\ell 8)\right]}{\pi(n+\ell 8)} \sin\left(\frac{\pi}{2}(n+\ell 8)\right) \{u[n]-u[n-8]\}$$

(c) Determine all 8 numerical values of the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \le k \le 7$. List the values clearly: $H_8(0) =?$, $H_8(1) =?$, $H_8(2) =?$, $H_8(3) =?$, $H_8(4) =?$, $H_8(5) =?$, $H_8(6) =?$, $H_8(7) =?$.

(d) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves.

$$x[n] = \sum_{k=0}^{7} e^{jk\frac{2\pi}{8}n} \{u[n]-u[n-8]\}$$

$y_8[n]$ is formed by computing $X_8(k)$ as an 8-pt DFT of $x[n]$, $H_8(k)$ as an 8-pt DFT of $h[n]$, and then $y_8[n]$ as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

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(a) $$h[n] = \frac{\sin\left[\pi\left(n+\frac{1}{2}\right)\right]}{\pi\left(n+\frac{1}{2}\right)} \qquad -\infty < n < \infty$$

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} h[n+\ell N]\left\{u[n]-u[n-N]\right\} \qquad n = 0, 1, \ldots, 8$$

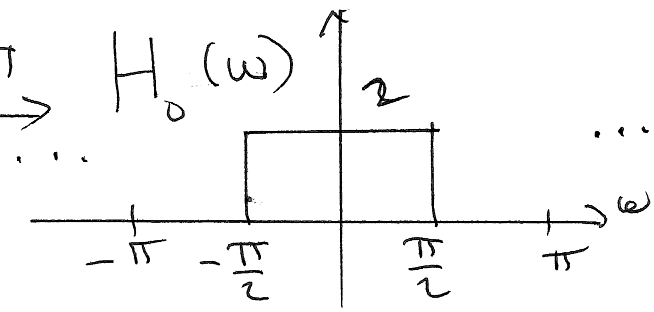
$N = 9$

Refer to part (a) of previous problem

$$H(\omega) = \left\{\frac{1}{2} H_0\left(\frac{\omega}{2}\right) + \frac{1}{2} e^{-j\pi} H_0\left(\frac{\omega-2\pi}{2}\right)\right\} e^{+j\frac{\omega}{2}}$$

where:

$$h_0[n] = \frac{\sin\left[\frac{\pi}{2}n\right]}{\frac{\pi}{2}n} \overset{DTFT}{\longleftrightarrow} H_0(\omega)$$



For $0 < \omega < \pi$

$$H(\omega) = e^{j\frac{\omega}{2}}$$

For $\pi < \omega < 2\pi$

$$H(\omega) = -e^{j\omega/2}$$

$k\frac{2\pi}{9} = \pi$

$k = \frac{9}{2}$ not integer so

$= 4.5$

we don't hit π luckily

we sample at $\omega_k = k\frac{2\pi}{9}$

$k = 0, 1, 2, 3, 4$

$$H_a(k) = e^{j\frac{1}{2}k\frac{2\pi}{9}} = e^{jk\frac{\pi}{9}}$$

$k = 5, 6, 7, 8$

$$H_a(k) = \cancel{+} e^{j\frac{1}{2}(\omega-2\pi)}\Big|_{\omega = k\frac{2\pi}{9}} = e^{j\frac{1}{2}\left(k\frac{2\pi}{9} - 2\pi\frac{9}{9}\right)} = e^{-jk\pi\frac{8}{9}}$$

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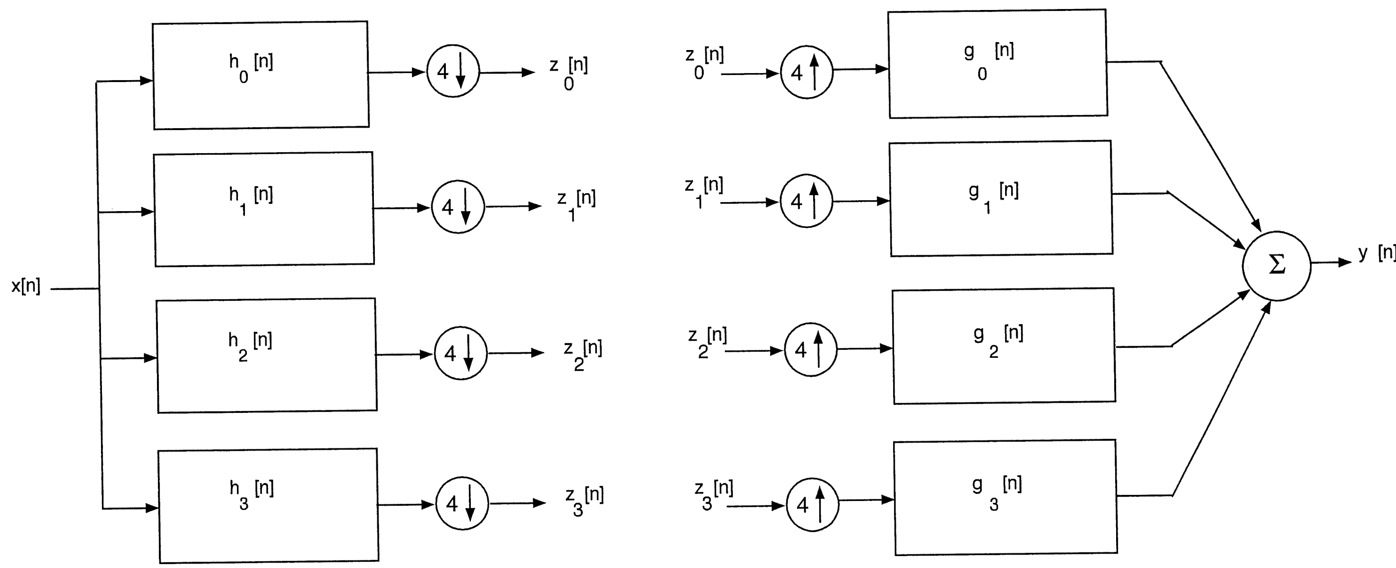

Figure 1(a). Analysis Filter Bank. Figure 1(b). Synthesis Filter Bank.

Figure 1. For Problem 4 on Next Page.

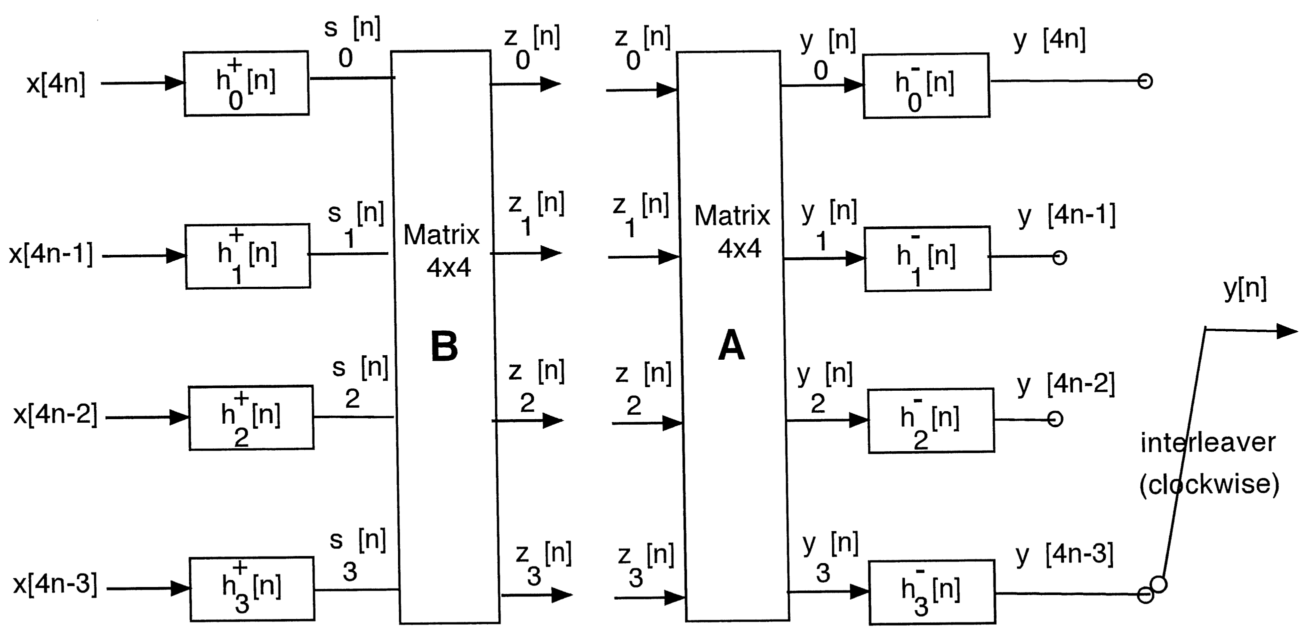

Figure 2. For Problem 4 on Next Page.

Problem 4. Consider the $M = 4$ channel Filter Bank in Figure 1 on the previous page. You are to determine whether it achieves Perfect Reconstruction with the following causal analysis filters, each of which is of length 4 and starts at $n = 0$.

$$h_0[n] = \{-1, 1, 1, 1\}$$

$$h_1[n] = \{1, -1, 1, 1\}$$

$$h_2[n] = \{1, 1, -1, 1\}$$

$$h_3[n] = \{1, 1, 1, -1\}$$

The corresponding synthesis filters are defined below. The four analysis filters and the four synthesis filters are all real-valued.

$$g_k[n] = h_k[-n], \quad k = 0, 1, 2, 3$$

In the space provided on the next three blank pages, you need to derive and clearly specify all filters and matrices in the computationally efficient implementation of this filter bank drawn in Figure 2 on the previous page. All answers are real-valued quantities.

NOTE: 1 The matrices **A** and **B** are **real-valued** matrices that have nothing to do with a 4-pt DFT matrix.

NOTE 2: After clearly specifying all the filters and matrices in Figure 2, determine if the Filter Bank achieves Perfect Reconstruction. Explain your answer.

On the analysis side, the efficient polyphase implementation of each subband channel involves

$$h_k^{(\ell)}[n] = h_k[4n + \ell] \qquad k = 0, 1, 2, 3 \qquad \ell = 0, 1, 2, 3$$

Since each filter is only of length 4, each polyphase component is only a single tap at $n = 0$!! meaning a scalar multiple of $\delta[n]$

$$h_k^{(\ell)}[0] = h_k[\ell]\,\delta[n]$$

For example:

$$h_k^{(2)}[0] = h_k[4 \cdot 0 + 2] = h_k[2]\,\delta[n]$$

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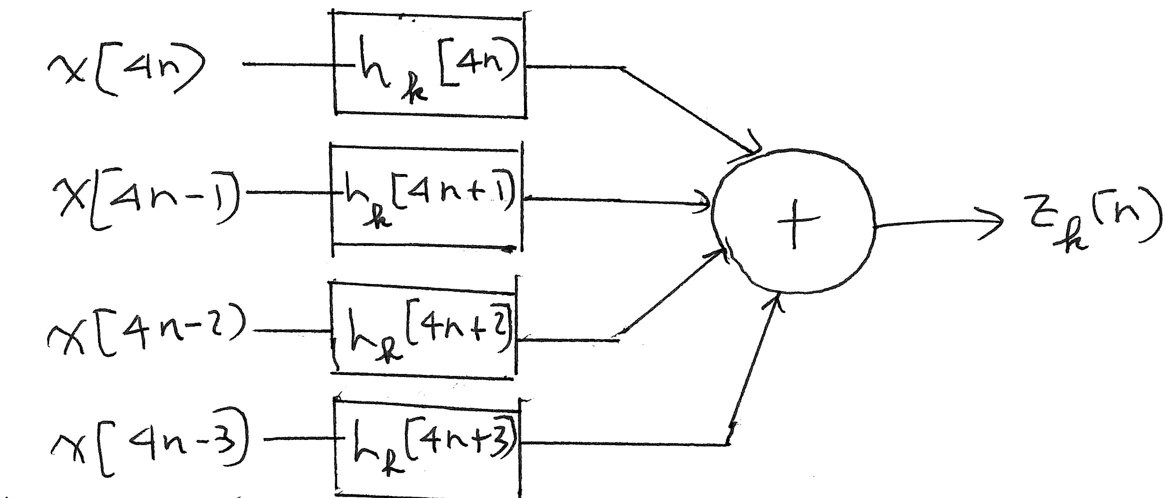
So, put all the scalars on $\delta[n]$ into the matrix B

because :



same inputs into each polyphase implementation of k-th channel

⟸ each is a scalar multiple of $\delta[n]$

Thus: $h_k^+[n] = \delta[n]$, $k = 0, 1, 2, 3$

and

$$\begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} = \overbrace{\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}}^{B \text{ matrix}} \begin{bmatrix} x[4n] \\ x[4n-1] \\ x[4n-2] \\ x[4n-3] \end{bmatrix}$$

k-th row is k-th filter impulse response

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On synthesis side, efficient polyphase implementation involves polyphase components

$g_\ell^{(\ell)}[n] = g_k[4n-\ell]$ where: $g_k[n] = h_k[-n]$

$= h_k[-(4n-\ell)] = h_k[-4n+\ell]$ } so same polyphase components as on analysis side

Again, because of the decimation by four with filters of only length 4, each polyphase component is of length 1 ⇒ meaning a scalar multiple of $\delta[n]$

Since, we are using a clockwise interleaver

$$\begin{bmatrix} y[4n] \\ y[4n-1] \\ y[4n-2] \\ y[4n-3] \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix}$$

$A = B^T$

BECAUSE in efficient polyphase implementation of zero inserts followed by filtering it's the SAME input into each polyphase filter

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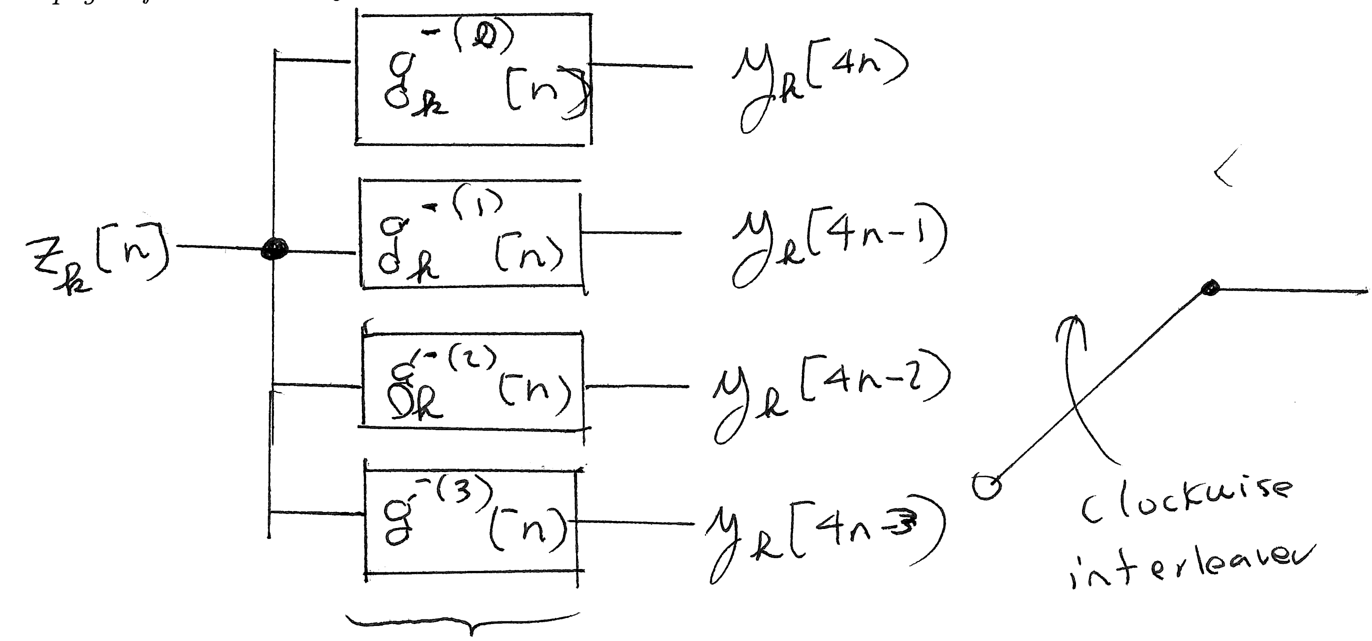

each of these is scalar multiple of $\delta[n)$

Since we eventually sum all the outputs on the synthesis side (of inefficient implementation) we can sum the "polyphase" components $y_k[4n-\ell)$ for each (ℓ) prior to interleaver

Since filters are mutually orthogonal)

$AB = B^T A = 4I$ and since

$h_k^+[n) = \delta[n)$ and $h_k^-[n) = \delta[n)$ $k=0,1,2,3$

this is a Perfect Reconstruction Filter Bank