Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Three 8.5 x 11 handwritten crib sheets allowed.
Calculators NOT allowed.
This test contains FIVE problems.
All work should be done in the proper space provided.

Good luck in this final exam!
Great having you in class this semester!
Have a joyous holiday season!
Problem 1. An OFDM signal is synthesized as a sum of four sinewaves of length 4 as

$$\tilde{x}[n] = \frac{1}{4} \sum_{k=0}^{3} b_k e^{j2\pi k n} \{ u[n] - u[n-4] \}$$

Each of the four values $b_k$ is one of the four values listed in the symbol alphabet below.

$$b_k \in \{1, 3, 5, 7\} \quad k = 0, 1, 2, 3$$

We know in advance that the signal we transmit will be convolved with a filter of length $L = 3$. Thus, we add a cyclic prefix of length 3 which effectively creates a sum of sinewaves of length 7 as prescribed below:

$$x[n] = \frac{1}{4} \sum_{k=0}^{3} b_k e^{j2\pi k n} \{ u[n+3] - u[n-4] \}$$

The sequence of length 7 above is convolved with the filter below of length 3

$$h[n] = \{4, 2, 1\} = 4\delta[n] + 2\delta[n-1] + 1\delta[n-2]$$

This ultimately yields the following sequence of length $7+3-1 = 9$

$$y[n] = x[n] * h[n] = \{-4-4j, -6-2j, -7+3j, 13+2j, 3-3j, -2-2j, -7+3j, -3+2j, -1+j\}$$

Your task is to determine the numerical values of each of the four symbols: $b_0$, $b_1$, $b_2$, and $b_3$. Explain and show all the steps in determining your answers. Please lay your work out nicely with logical ordering. Your work will be more important than your final answers.
This page left intentionally blank for student work.
Problem 2.

(a) Determine the autocorrelation $r_{x_0x_0}[\ell]$ of the length-4 sequence $x_0[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

\[ x_0[n] = u[n] - u[n - 4] = \{1, 1, 1, 1\} \]

(b) Determine the autocorrelation $r_{x_1x_1}[\ell]$ of the length-4 sequence $x_1[n]$ below. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

\[ x_1[n] = e^{j \left( \frac{\pi n}{\sqrt{2}} + \frac{\pi}{\sqrt{2}} \right)} \{u[n] - u[n - 4]\} \]

(c) Determine the autocorrelation $r_{x_2x_2}[\ell]$ of the length-4 sequence $x_2[n]$ below. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

\[ x_2[n] = e^{j\pi(n-2)} \{u[n - 2] - u[n - 6]\} \]

(d) Determine the autocorrelation $r_{x_3x_3}[\ell]$ of the length-4 sequence $x_3[n]$ below. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

\[ x_3[n] = e^{-j \left( \frac{\pi n}{\sqrt{2}} + \frac{\pi}{\sqrt{2}} \right)} \{u[n] - u[n - 4]\} \]

(e) Sum your answers to parts (a) thru (d) to form the sum below. Do a stem plot of $r_{xx}[\ell]$.

\[ r_{xx}[\ell] = r_{x_0x_0}[\ell] + r_{x_1x_1}[\ell] + r_{x_2x_2}[\ell] + r_{x_3x_3}[\ell] \]
Problem 3. Let \( x[n] \) be of length \( L = 8 \), i.e., \( x[n] = 0 \) for \( n < 0 \) and \( n \geq 8 \) and \( h[n] \) be of length \( M = 5 \), i.e., \( h[n] = 0 \) for \( n < 0 \) and \( n \geq 5 \). Let \( X_8(k) \) and \( H_8(k) \) denote 8-point DFT’s of \( x[n] \) and \( h[n] \), respectively. The 8-point inverse DFT of the product \( Y_8(k) = X_8(k)H_8(k) \), denoted \( y_8[n] \), produces the following values:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_8[n] )</td>
<td>75</td>
<td>58</td>
<td>49</td>
<td>48</td>
<td>55</td>
<td>70</td>
<td>85</td>
<td>100</td>
</tr>
</tbody>
</table>

Let \( X_{10}(k) \) and \( H_{10}(k) \) denote the 10-point DFT’s of the aforementioned sequences \( x[n] \) and \( h[n] \). The 10-point inverse DFT of the product \( Y_{10}(k) = X_{10}(k)H_{10}(k) \), denoted \( y_{10}[n] \), produces the following values:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_{10}[n] )</td>
<td>28</td>
<td>22</td>
<td>26</td>
<td>40</td>
<td>55</td>
<td>70</td>
<td>85</td>
<td>100</td>
<td>70</td>
<td>44</td>
</tr>
</tbody>
</table>

Given \( y_8[n] \) and \( y_{10}[n] \), find the linear convolution of \( x[n] \) and \( h[n] \), i.e., list all of the numerical values of \( y[n] = x[n] * h[n] \). Hint: all the values are positive, integers.
This page left intentionally blank for student work.
Problem 4. (a)

(a) Consider $h_1[n]$ and $h_2[n]$ to be two distinct all-pass filters ($p_1 \neq p_2$) with respective impulse responses below; $p_1$ and $p_2$ are both real-valued with absolute value less than 1. Is the sum $h[n] = h_1[n] + h_2[n]$ an all-pass filter for any and all values of $p_1$ and $p_2$ ($p_1 \neq p_2$)? Explain your answer. Your explanation is much more important than your answer.

\[ h_1[n] = \frac{1}{p_1} \left\{ \delta[n] + (p_1^2 - 1)p_1^n u[n] \right\} \]  

\[ h_2[n] = \frac{1}{p_2} \left\{ \delta[n] + (p_2^2 - 1)p_2^n u[n] \right\} \]
Problem 4. (b)

(b) Consider $h_1[n]$ and $h_2[n]$ to be two distinct all-pass filters ($p_1 \neq p_2$) with respective impulse responses below; $p_1$ and $p_2$ are both real-valued with absolute value less than 1. Is the convolution $h[n] = h_1[n] * h_2[n]$ an all-pass filter for any and all values of $p_1$ and $p_2$? Explain your answer. Your explanation is much more important than your answer.

$$h_1[n] = \frac{1}{p_1} \left\{ \delta[n] + (p_1^2 - 1)p_1^n u[n] \right\} \quad (3)$$

$$h_2[n] = \frac{1}{p_2} \left\{ \delta[n] + (p_2^2 - 1)p_2^n u[n] \right\} \quad (4)$$
(c) Consider $h[n]$ to be an all-pass filter with respective impulse response below.

$$h[n] = \frac{1}{p} \left\{ \delta[n] + (p^2 - 1)p^n u[n] \right\} \quad (5)$$

Is the product

$$g[n] = e^{j\omega_0 n} h[n]$$

an all-pass filter for any and all values of the frequency $\omega_0$? Explain your answer. Your explanation is much more important than your answer.
Problem 5.

(a) Consider sampling the DTFT below at $N = 8$ equi-spaced frequencies over $0 \leq \omega < 2\pi$.

$$X(\omega) = \frac{1}{1 - 0.8e^{-j\omega}} + \frac{1}{1 - 0.9e^{-j\omega}}$$

That is, let $X_8(k) = X(2\pi k/8)$, $k = 0, 1, \ldots, 7$, where $X(\omega)$ is defined above. Now, compute $x_8[n]$ as the 8-point Inverse DFT of $X_8(k)$. Determine a closed-form (“plug-and-chug”) expression for $x_8[n]$ that prescribes the value of $x_8[n]$ for $n = 0, 1, \ldots, 7$. Show all work. **Note:** A **closed-form** expression contains NO summations and it is NOT a listing of numbers. **Hint:**

$$\frac{1}{1 - (.9)^8} = 1.7558 \quad \frac{1}{1 - (.8)^8} = 1.2016$$

(b) Show how your answer changes when everything is the same as defined in part (a) except that the DTFT is now defined as below.

$$X(\omega) = \frac{8}{1 - 0.8e^{-j\omega}} - \frac{9}{1 - 0.9e^{-j\omega}}$$

(c) Show how your answer changes when everything is the same as defined in part (a) except that the DTFT is now defined as below.

$$X(\omega) = \frac{e^{j2\omega}}{1 - 0.8e^{-j\omega}} + \frac{e^{j4\omega}}{1 - 0.9e^{-j\omega}}$$

(d) Show how your answer changes when everything is the same as defined in part (a) except that the DTFT is now defined below. (That is, the two terms are now multiplied rather than being summed as in part (a).) Show all work.

$$X(\omega) = \frac{1}{1 - 0.8e^{-j\omega}} \cdot \frac{1}{1 - 0.9e^{-j\omega}}$$
This page left intentionally blank for student work for student work.
This page left intentionally blank for student work for student work.
This page left intentionally blank for student work for student work.