

**ECE538**  
**Digital Signal Processing I**

**Final Exam**

**Fall 2017**  
**14 December 2017**

**Cover Sheet**

Test Duration: 2 hours.

Open Book but Closed Notes.

Three double-sided 8.5 x 11 crib sheets allowed

This test contains **five** problems.

You must show all work for each problem to receive full credit.

<b>No.</b>	<b>Topic(s) of Problem</b>	<b>Points</b>
1.	OFDM	20
2.	Noble's Identities	20
3.	Perfect Reconstruction Filter Bank	20
4.	DFT's and Time-Domain Aliasing	20
5.	Z-Transform, Difference Eqns., Bilinear Transform	20

**Problem 1.** An OFDM signal is synthesized as a sum of four sinewaves of length 4 as

$$\tilde{x}[n] = \frac{1}{4} \sum_{k=0}^3 b_k e^{j2\pi \frac{k}{4}n} \{u[n] - u[n-4]\}$$

Each of the four values  $b_k$  is one of the four QPSK values listed in the symbol alphabet below.

$$b_k \in \{1 + j, 1 - j, -1 + j, -1 - j\} \quad k = 0, 1, 2, 3$$

We know in advance that the signal we transmit will be convolved with a filter of length  $L = 3$ . Thus, we add a cyclic prefix of length 3 which effectively creates a sum of sinewaves of length 7 as prescribed below:

$$x[n] = \frac{1}{4} \sum_{k=0}^3 b_k e^{j2\pi \frac{k}{4}n} \{u[n+3] - u[n-4]\}$$

The sequence of length 7 above is convolved with the filter below of length 3

$$h[n] = \{8, 4, 2\} = 8\delta[n] + 4\delta[n-1] + 2\delta[n-2]$$

This ultimately yields the following sequence of length  $7+3-1 = 9$

$$y[n] = x[n] * h[n] = \{4 + 4j, -2 + 6j, 3 + 7j, 5 - j, 7 + 3j, -1 + 5j, 3 + 7j, 1 + 3j, 1 + j\}$$

Your task is to determine the numerical values of each of the four symbols:  $b_0$ ,  $b_1$ ,  $b_2$ , and  $b_3$ . Explain and show all the steps in determining your answers. Please lay your work out nicely with logical ordering. Your work will be more important than your final answers.

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**Problem 2.** [20 points] Determine the impulse response  $h[n]$  in Figure 2.1(b) so that the I/O relationship of the system in Figure 2.1(b) is exactly the same as the I/O relationship of the system in Figure 2.1(a). Plot the magnitude AND the phase (two separate plots) of the DTFT of  $h[n]$  over  $-\pi < \omega < \pi$ . *Hint:* Analyze the system of Figure 2.1(a) in the frequency domain using one of Noble's Identities.

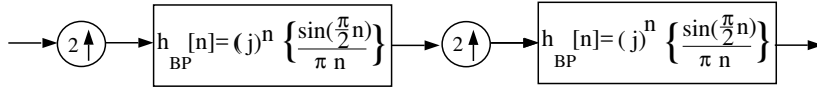


Figure 2.1(a).

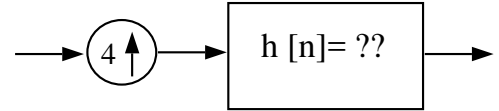


Figure 2.1(b).

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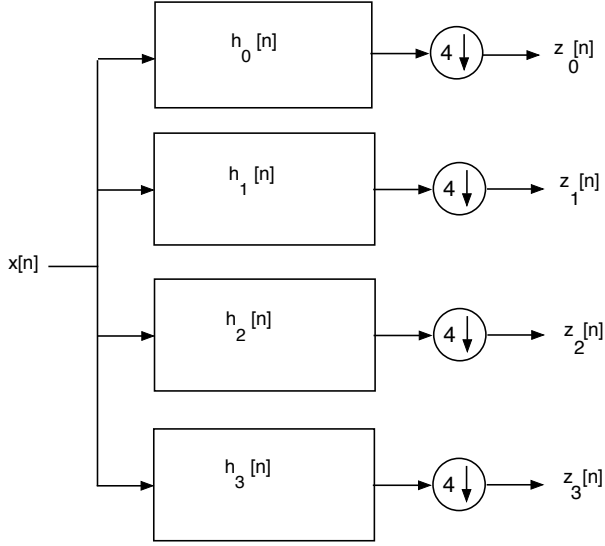


Figure 1(a). Analysis Filter Bank.

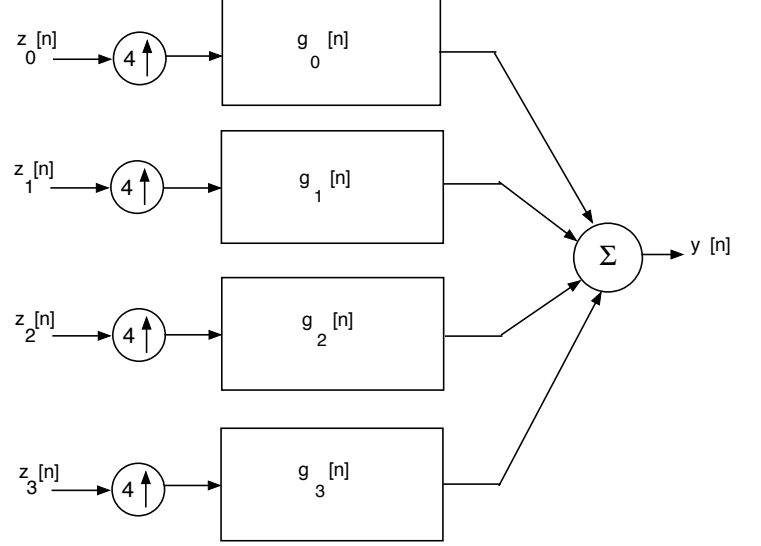


Figure 1(b). Synthesis Filter Bank.

Figure 3.1. For Problem 3 on Next Page.

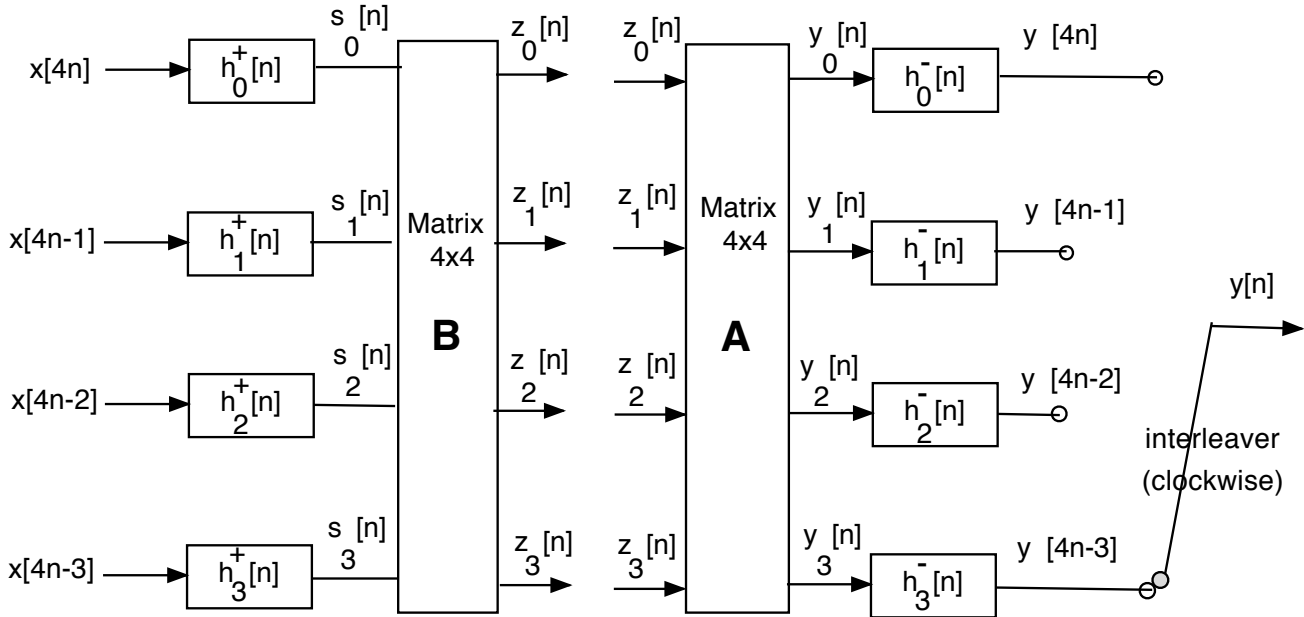


Figure 3.2. For Problem 3 on Next Page.

**Problem 3.** Consider the  $M = 4$  channel Filter Bank in Figure 3.1 on the previous page. You are to determine an efficient implementation of the entire structure having the form in Figure 3.2. Then determine whether it achieves Perfect Reconstruction with the following **causal** analysis filters, each of which is of length 4 and starts at  $n = 0$ . Achieving Perfect Reconstruction means the output is the same as the input except for possibly an amplitude-scaling and an integer-delay.

$$h_0[n] = \{1, -2, -1, -2\}$$

$$h_1[n] = \{2, 1, 2, -1\}$$

$$h_2[n] = \{1, -2, 1, 2\}$$

$$h_3[n] = \{2, 1, -2, 1\}$$

The first element in each vector corresponds to  $n = 0$ . The corresponding synthesis filters are defined below. The four analysis filters and the four synthesis filters are all real-valued.

- (a) Draw the efficient implementation of the top branch of the analysis side where the  $h_0[n]$  is followed by decimation by a factor of 4. The inputs for the efficient implementation should be  $x[4n]$ ,  $x[4n - 1]$ ,  $x[4n - 2]$ ,  $x[4n - 3]$ .
- (b) Draw the efficient implementation of the top branch of the synthesis side where zero-inserts is followed filtering by  $g_0[n]$  followed by decimation by a factor of 4. The outputs for the efficient implementation should be  $y[4n]$ ,  $y[4n - 1]$ ,  $y[4n - 2]$ ,  $y[4n - 3]$  and the commutator for the interleaver should operate clockwise.
- (c) Since all of the analysis and synthesis filters are each of length 4, the polyphase components are each of length 1. Thus, we can set

$$h_0^+[n] = \delta[n] \quad h_1^+[n] = \delta[n] \quad h_2^+[n] = \delta[n] \quad h_3^+[n] = \delta[n]$$

$$h_0^-[n] = \delta[n] \quad h_1^-[n] = \delta[n] \quad h_2^-[n] = \delta[n] \quad h_3^-[n] = \delta[n]$$

With this in mind, the matrices **B** and **A** operate as follows.

$$\begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} x[4n] \\ x[4n - 1] \\ x[4n - 2] \\ x[4n - 3] \end{bmatrix} = \begin{bmatrix} y[4n] \\ y[4n - 1] \\ y[4n - 2] \\ y[4n - 3] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} \quad (1)$$

Determine the matrices **A** and **B** in this efficient implementation.

- (d) Determine if the Filter Bank achieves Perfect Reconstruction. Explain your answer.

**NOTE: 1** The matrices **A** and **B** are **real-valued** matrices that have nothing to do with a 4-pt DFT matrix.

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Since each of the analysis filters is only of length  $M=4$ , which is the no. of subbands, we have:

$$\begin{aligned}
 Z_m[n] &= \sum_{k=0}^3 h_m[k] x[n-k] \quad \left( n=4n \right) \\
 &= \sum_{k=0}^3 h_m[k] x[4n-k] \\
 &= \begin{bmatrix} h_m[0] & h_m[1] & h_m[2] & h_m[3] \end{bmatrix} \begin{bmatrix} x[4n] \\ x[4n-1] \\ x[4n-2] \\ x[4n-3] \end{bmatrix}
 \end{aligned}$$

$m=0,1,2,3$

Thus, it is easy to see that

$$B = \begin{bmatrix} h_0[0] & h_0[1] & h_0[2] & h_0[3] \\ h_1[0] & h_1[1] & h_1[2] & h_1[3] \\ h_2[0] & h_2[1] & h_2[2] & h_2[3] \\ h_3[0] & h_3[1] & h_3[2] & h_3[3] \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & -2 \\ 2 & 1 & 2 & -1 \\ 1 & -2 & 1 & 2 \\ 2 & 1 & -2 & 1 \end{bmatrix}$$

Clearly, Perfect Reconstruction is achieved with  $A = B^{-1} = B^T$  since rows of  $B$  are orthogonal

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -2 & 1 & -2 & 1 \\ -1 & 2 & 1 & -2 \\ -2 & -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} g_0[0] & g_0[1] & g_0[2] & g_0[3] \\ g_1[0] & g_1[1] & g_1[2] & g_1[3] \\ g_2[0] & g_2[1] & g_2[2] & g_2[3] \\ g_3[0] & g_3[1] & g_3[2] & g_3[3] \end{bmatrix}$$

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**Problem 4.** Let  $x[n]$  be of length  $L = 4$ , i.e.,  $x[n] = 0$  for  $n < 0$  and  $n \geq 4$  and  $h[n]$  be of length  $M = 5$ , i.e.,  $h[n] = 0$  for  $n < 0$  and  $n \geq 5$ . Let  $X_6(k)$  and  $H_6(k)$  denote 6-point DFT's of  $x[n]$  and  $h[n]$ , respectively. The 6-point inverse DFT of the product  $Y_6(k) = X_6(k)H_6(k)$ , denoted  $y_6[n]$ , produces the following values:

n	0	1	2	3	4	5
$y_6[n]$	3	3	3	4	4	3

Let  $X_5(k)$  and  $H_5(k)$  denote the 5-point DFT's of the aforementioned sequences  $x[n]$  and  $h[n]$ . The 5-point inverse DFT of the product  $Y_5(k) = X_5(k)H_5(k)$ , denoted  $y_5[n]$ , produces the following values:

n	0	1	2	3	4
$y_5[n]$	4	4	4	4	4

Given  $y_6[n]$  and  $y_5[n]$ , find the **linear** convolution of  $x[n]$  and  $h[n]$ , i. e., list all of the numerical values of  $y[n] = x[n] * h[n]$ .

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**Problem 5.** A second-order digital filter is to be designed from an analog filter having two poles in the s-plane at  $-1 + 2j$  and  $-1 - 2j$  and two zeros at  $j$  and  $-j$ , via the bilinear transformation method characterized by the mapping

$$s = \frac{z - 1}{z + 1}$$

- (a) Is the resulting digital filter (BIBO) stable? Briefly explain why or why not.
- (b) Denote the frequency response of the resulting digital filter as  $H(\omega)$  (the DTFT of its impulse response). You are given that in the range  $0 < \omega < \pi$ , there is only one value of  $\omega$  for which  $H(\omega) = 0$ . Determine that value of  $\omega$ .
- (c) Draw a pole-zero diagram for the resulting **digital** filter. Give the exact locations of the poles and zeros of the digital filter in the z-plane.
- (d) Plot the magnitude of the DTFT of the resulting digital filter,  $|H(\omega)|$ , over  $-\pi < \omega < \pi$ . You are given that  $H(0) = 0.8$ . Be sure to indicate any frequency for which  $|H(\omega)| = 0$ . Also, specifically note the numerical value of  $|H(\omega)|$  for  $\omega = \frac{\pi}{2}$  and  $\omega = \pi$ .
- (e) Determine the difference equation for the resulting digital filter.

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$$s = \frac{z-1}{z+1} \Rightarrow s(z+1) = z-1 \Rightarrow sz + s = z-1$$

$$\Rightarrow z(s-1) = 1+s \Rightarrow z = \frac{1+s}{1-s}$$

(a) poles of analog filter are in LHP  $\Rightarrow$   
 bilinear transform guarantees poles mapped  
 to values inside unit circle in  $z$ -plane  
 $\Rightarrow$  STABLE!

(b) zero at  $s=j$  on imaginary axis mapped  
 to zero on unit circle at

$$z = \frac{1+s}{1-s} \Big|_{s=j} = \frac{1+j}{1-j} = \frac{\sqrt{2} e^{j\frac{\pi}{4}}}{\sqrt{2} e^{-j\frac{\pi}{4}}} = e^{j\frac{\pi}{2}} = j$$

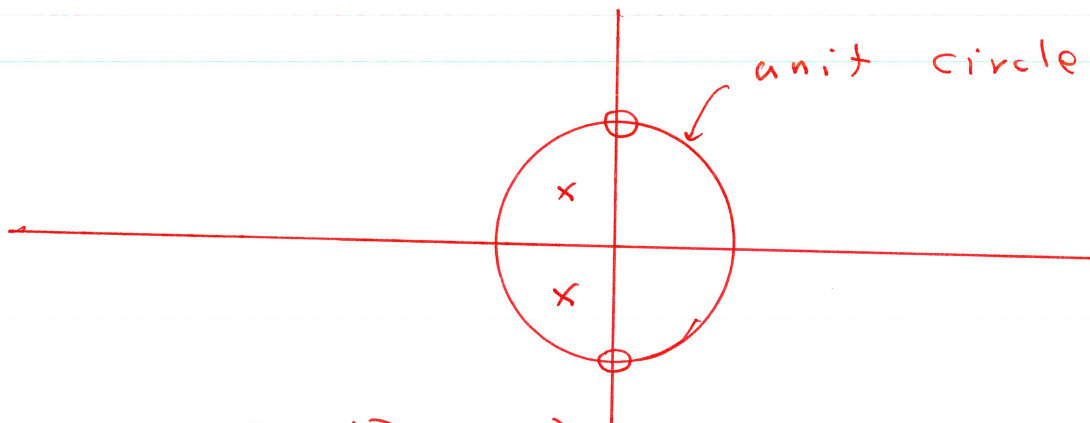
$$\Rightarrow H\left(\frac{\pi}{2}\right) = 0$$

(c)  $-1+2j$  mapped to  $\frac{1+(-1+2j)}{1-(-1+2j)} = \frac{2j}{2-2j}$

$$= \frac{j}{1-j} \frac{(1+j)}{(1+j)} = \frac{-1+j}{2} = \frac{\sqrt{2} e^{j\frac{3\pi}{4}}}{2} = \frac{1}{\sqrt{2}} e^{j\frac{3\pi}{4}}$$

$-1-2j$  mapped to  $\frac{1}{\sqrt{2}} e^{-j\frac{3\pi}{4}}$

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$$(d) H(z) = \frac{(z-j)(z+j)G}{(z-e^{j\frac{3\pi}{4}})(z-e^{-j\frac{3\pi}{4}})} = \frac{G(z^2+1)}{z^2+z+\frac{1}{2}}$$

$$H(z) \Big|_{z=1} = .8 = \frac{G(1+1)}{1+1+\frac{1}{2}} = \frac{2G}{\frac{5}{2}} = \frac{4}{5}G$$

$$\omega=0$$

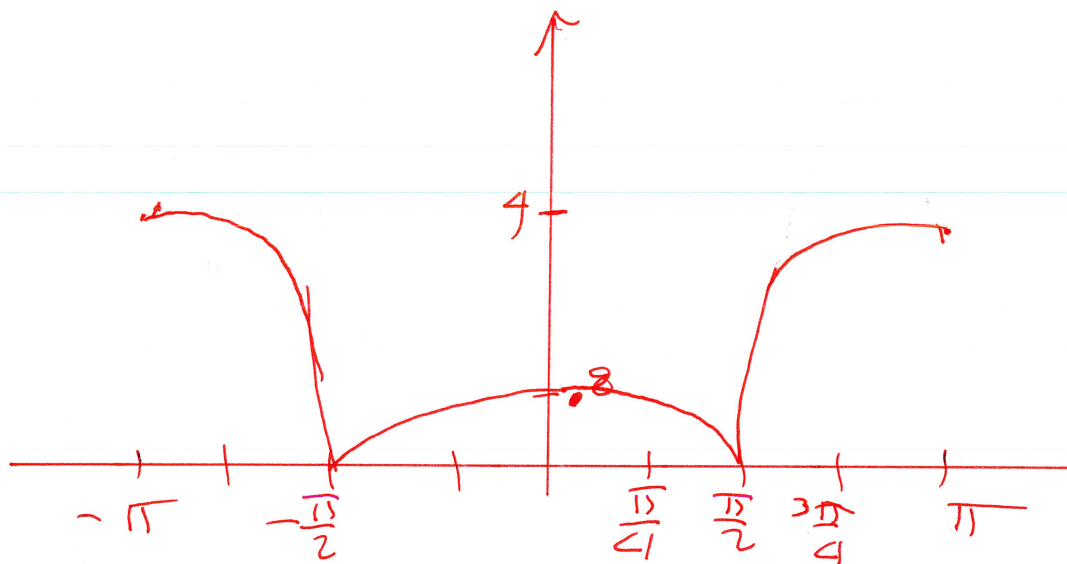
$$G=1$$

$$H\left(\frac{\pi}{2}\right) = 0 = H\left(-\frac{\pi}{2}\right)$$

$$H(\pi) = H(z) \Big|_{z=-1} = \frac{(-1)^2+1}{(-1)^2-1+\frac{1}{2}} = \frac{2}{1/2} = 4$$

$$H\left(\frac{3\pi}{4}\right) = \frac{e^{j2\frac{3\pi}{4}}+1}{e^{j2\frac{3\pi}{4}}+e^{j\frac{3\pi}{4}}+\frac{1}{2}} = \frac{-j+1}{-j+e^{j\frac{3\pi}{4}}+\frac{1}{2}}$$

$$\left|H\left(\frac{3\pi}{4}\right)\right| \approx 3.94 \approx 4$$



$$(e) \frac{Y(z)}{X(z)} = \frac{z^2 + 1}{z^2 + z + \frac{1}{2}} \equiv \frac{1 + z^{-2}}{1 + z^{-1} + \frac{1}{2} z^{-2}}$$

$$y[n] + y[n-1] + \frac{1}{2} y[n-2] = x[n] + x[n-2]$$

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