NAME: **Final Exam**
ECE 538 Digital Signal Processing I **14 Dec. 2011**
Room: WTHR 172 **8-10 am**

Cover Sheet

WRITE YOUR NAME ON THIS COVER SHEET

Test Duration: 120 minutes.

Open Book but Closed Notes.

Calculators NOT allowed.

All work should be done in the space provided.

Problem 1. This problem is about determining whether the four-band analysis filter bank and corresponding four-band synthesis filter bank in Figure 1 achieves Perfect Reconstruction when the respective impulse responses for the four filters

$$f_k[n] = e^{j(k\frac{2\pi}{4})n} h_{LP}[n], \quad k = 0, 1, 2, 3$$

are defined in terms of the lowpass filter with impulse response below, which is of length 4.

$$h_{LP}[n] = u[n] - u[n-4]. \tag{1}$$

This question is more easily answered examining the efficient version of the overall filter bank in Figure 2.

(a) Plot the magnitude of the DTFT $H_{LP}(\omega)$ of $h_{LP}[n]$ over $-\pi < \omega < \pi$. Will there be aliasing present in each of the outputs $z_k[n]$?

(b) The four polyphase components of $h_{LP}[n]$ are defined as

$$h_\ell^+[n] = h_{LP}[4n + \ell], \quad \ell = 0, 1, 2, 3. \tag{2}$$

Do a separate stem plot (time-domain) of each of the four impulse responses $h_\ell^+[n]$, $\ell = 0, 1, 2, 3$. Are they all the same?

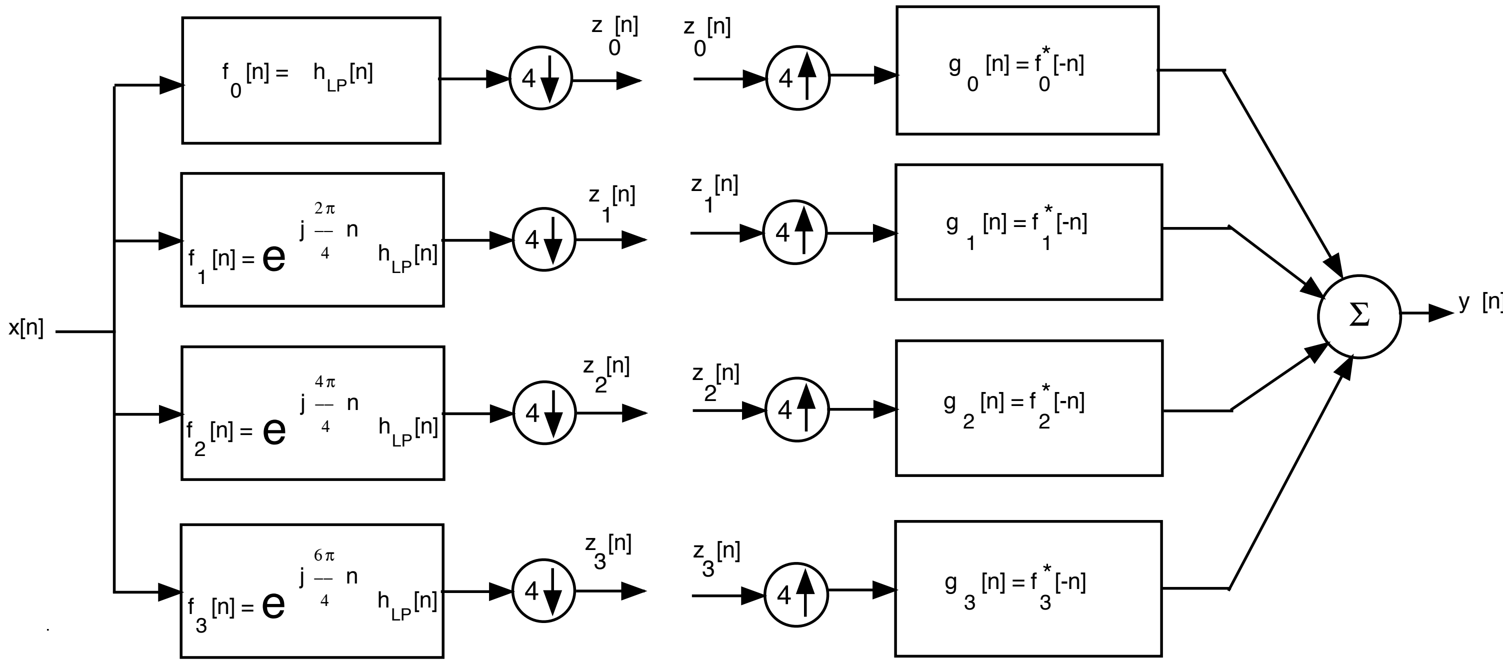

Figure 1.

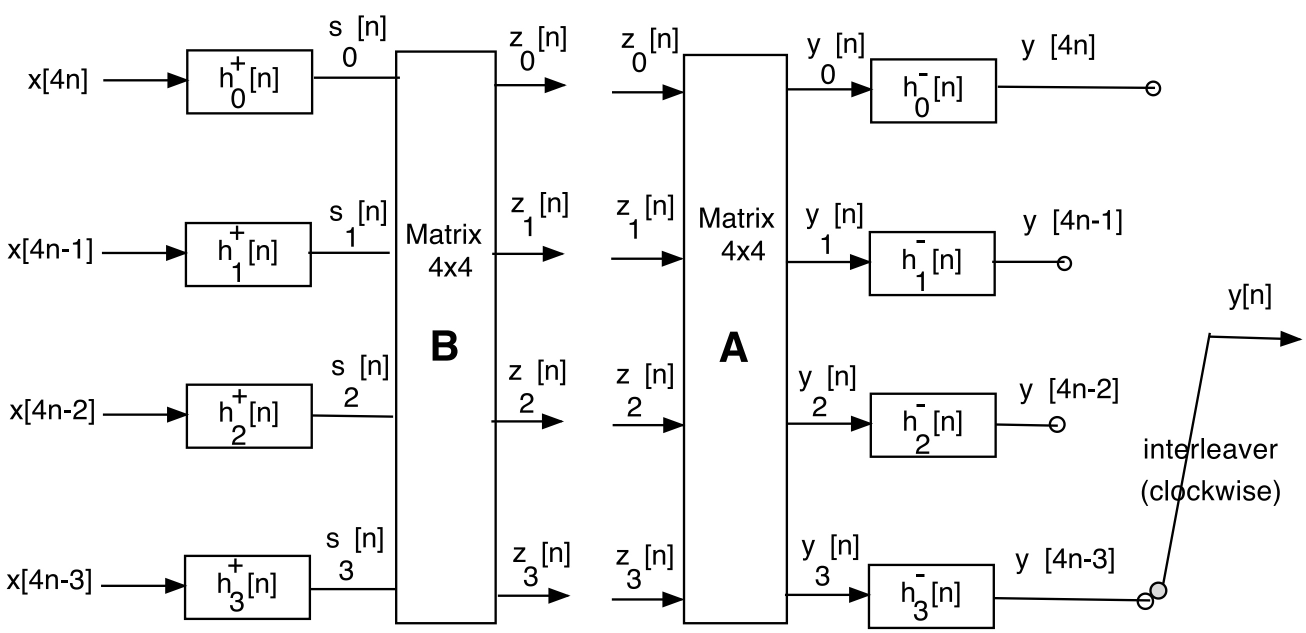

Figure 2.

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(c) The digital subbanding on the left hand side of Figure 1 is alternatively effected more efficiently with the polyphase filters via the block diagram on the left side of Figure 2, where the respective input signals are defined as below. Fill in the values of $\beta_{k\ell}$ in the table below so that the outputs, $z_k[n]$, $k = 0, 1, 2, 3$, at the output of the left hand side in Figure 2 are exactly the same as the outputs $z_k[n]$ of the left hand side in Figure 1. Show all work.

$$\begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \beta_{11} & \beta_{12} & \beta_{13} \\ 1 & \beta_{21} & \beta_{22} & \beta_{23} \\ 1 & \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{bmatrix} s_0[n] \\ s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix} \quad \Rightarrow \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \beta_{11} & \beta_{12} & \beta_{13} \\ 1 & \beta_{21} & \beta_{22} & \beta_{23} \\ 1 & \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \tag{3}$$

$\beta_{k\ell}$	ℓ =1	ℓ =2	ℓ =3
$k = 1$			
$k = 2$			
$k = 3$			

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(d) The zero inserts followed by filtering is done more efficiently in a collective manner. First, note, in order to match the inputs with the outputs, we effect the efficient implementation of "L zero inserts followed by filtering" in an alternative fashion relative to the way it was originally derived in class. In this case, we interleave in a **clockwise** fashion the following outputs below for $\ell = 0, 1, 2, 3$.

$$\begin{aligned} y[Ln-\ell] &= x[n] * h[Ln-\ell] \qquad (4) \\ y[Ln-\ell] &= x[n] * h_\ell^-[n] \qquad (5) \\ & \qquad (6) \end{aligned}$$

where $h_\ell^-[n] = h[Ln-\ell]$, $\ell = 0, 1, ..., L-1$.

Thus, the polyphase components of interest for this part of the problem are

$$h_\ell^-[n] = g_0[4n-\ell], \quad \ell = 0, 1, 2, 3, \qquad (7)$$

where $g_0[n] = h_{LP}[-n]$ with $h_{LP}[n] = u[n] - u[n-4]$ as before. It follows that

$$h_\ell^-[n] = h_{LP}[\ell - 4n], \quad \ell = 0, 1, 2, 3, \qquad (8)$$

where, again, $h_{LP}[n] = u[n] - u[n-4]$ is only of length 4.

With this in mind, determine the value of each of the amplitude coefficients $\alpha_{\ell k}$, $k = 1, 2, 3$, $\ell = 1, 2, 3$ so that the interleaved output $y[n]$ in Figure 2 is the SAME as the overall output in Figure 1. Show all work on the next page but input your final answers in the table below.

$$\begin{bmatrix} y_0[n] \\ y_1[n] \\ y_2[n] \\ y_3[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 1 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 1 & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} \quad \Rightarrow \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 1 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 1 & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \qquad (9)$$

$\alpha_{\ell k}$	k =1	k =2	k =3
$\ell = 1$			
$\ell = 2$			
$\ell = 3$			

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(e) For part (c), we defined the following four polyphase components of $h_{LP}[n]$

$$h_\ell^-[n] = g_0[4n-\ell], \quad \ell = 0, 1, 2, 3. \tag{10}$$

where $g_0[n] = h_{LP}[-n]$ with $h_{LP}[n] = u[n] - u[n-4]$. Do a separate stem plot (time-domain) of each of the four impulse responses $h_\ell^-[n]$, $\ell = 0, 1, 2, 3$. Are they all the same?

(f) Does $y[n] = 4x[n]$? You need to either prove or disprove $y[n] = 4x[n]$ with all that you've developed so far for parts (a) through (e).
Hint: assess if $y[4n-\ell] = 4x[4n-\ell]$, for $\ell = 0, 1, 2, 3$.

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Problem 2. [20 points]

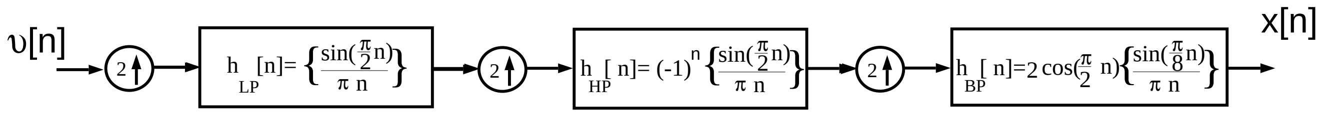

Figure 3(a).

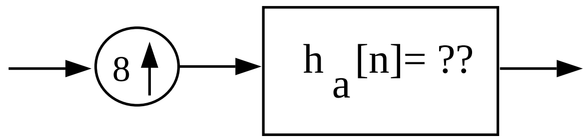

Figure 3(b).

(a) Determine the impulse response $h_a[n]$ in Figure 3(b) so that the I/O relationship of the system in Figure 3(b) is exactly the same as the I/O relationship of the system in Figure 3(a). Plot the magnitude of the DTFT of $h_a[n]$ over $-\pi < \omega < \pi$.

(b) Determine the autocorrelation sequence $r_{\nu\nu}[\ell]$ for the input $\nu[n]$ below with $p = \frac{1}{2}$.

$$\nu[n] = \frac{1}{p}\left\{\delta[n] \ + (p^2 - 1)p^n u[n]\right\} \tag{11}$$

(c) Determine and plot the DTFT of the **output** autocorrelation sequence $r_{xx}[\ell]$ over $-\pi \leq \omega \leq \pi$.

$$S_{xx}(\omega) = \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell]e^{-j\omega\ell}$$

(d) Determine an expression for the autocorrelation sequence $r_{xx}[\ell]$ for the output $x[n]$.

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Problem 3.
A discrete-time system of the form $y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n] + b_1 x[n-1]$ has a corresponding causal impulse response of the form

$$h[n] = c_1 \, (p_1)^n \, u[n] \; + \; c_2 \, (p_2)^n \, u[n]$$

where $u[n]$ is the discrete-time unit step sequence. You are given the first four values of this causal **infinite length** impulse response (IIR) below.

$$\begin{aligned} h[0] &= 35 \\ h[1] &= 13 \\ h[2] &= 5 \\ h[3] &= 2 \end{aligned}$$

(a) Determine the respective numerical values of the coefficients a_1 and a_2.

(b) Determine the respective numerical values of the two poles p_1 and p_2.

(c) Determine the respective numerical values of the values of c_1 and c_2 in the expression for the impulse response $h[n]$ above.

(d) Draw a block diagram for this system in terms of two first-order systems in parallel, and thus involving only TWO unit delay blocks.

HINT: Set up equation for the impulse response and then examine the recursion for $n > 1$.

NOTE: Trying to set up four equations in four unknowns in the straightforward manner that first comes to mind leads to a nonlinear system of equations involving quadratic and cubic nonlinear equations in the unknowns. This is NOT the way to solve the problem and does not use any signal processing knowledge or background. You will not receive any points for taking this approach.

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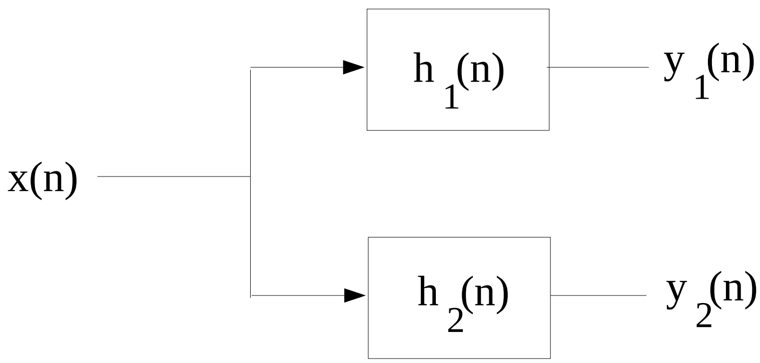
Problem 4.



Consider the system above where the causal signal $x(n)$ is simultaneously input to two different causal FIR filters of length $M = 2$ with respective impulses responses $h_1(n)$ and $h_2(n)$. That is, both $h_1(n)$ and $h_2(n)$ are only nonzero for $n = 0$ and $n = 1$. We don't know anything about the input signal $x(n)$ except that it is causal. Yet, it is possible to determine $h_1(n)$ and $h_2(n)$ (to within a scalar multiple) given the respective outputs $y_1(n)$ and $y_2(n)$.

(a) Show that $y_1(n) * h_2(n) = y_2(n) * h_1(n)$, which may be alternatively expressed as

$$y_1(n) * h_2(n) - y_2(n) * h_1(n) = 0$$

(b) Exploiting this relationship and arbitrarily assigning $h_1(0) = 1$ without loss of generality, determine the numerical values of $h_1(1)$, $h_2(0)$, and $h_2(1)$ given the following output values.

$$\begin{array}{cccc} y_1(0) = 1 & y_1(1) = 4 & y_1(2) = 7 & y_1(3) = 10 \\ y_2(0) = 3 & y_2(1) = 5 & y_2(2) = 7 & y_2(3) = 9 \end{array}$$

Note: This is not all of the output values for both $y_1(n)$ and $y_2(n)$. However, it is enough information to determine both $h_1(n)$ and $h_2(n)$ which, again, are only nonzero for $n = 0$ and $n = 1$. Again, you are given $h_1(0) = 1$. SHOW ALL WORK.

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