Digital Signal Processing I **Final Exam** **Fall 2010**
ECE538 **13 Dec.. 2010**

Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains **FOUR** problems.
All work should be done in the blue book provided.
Do **not** return this test sheet, just return your blue book.

Problem 1.

(a) Let $x[n]$ and $y[n]$ be real-valued sequences both of which are even-symmetric: $x[n] = x[-n]$ and $y[n] = y[-n]$. Under these conditions, prove that $r_{xy}[\ell] = r_{yx}[\ell]$ for all ℓ.

(b) Express the autocorrelation sequence $r_{zz}[\ell]$ for the complex-valued signal $z[n] = x[n] + jy[n]$ where $x[n]$ and $y[n]$ are real-valued sequences, in terms of $r_{xx}[\ell]$, $r_{xy}[\ell]$, $r_{yx}[\ell]$, and $r_{yy}[\ell]$.

(c) Determine a closed-form expression for the autocorrelation sequence $r_{xx}[\ell]$ for the signal $x[n]$ below.

$$x[n] = \left\{ \frac{\sin(\frac{\pi}{2}n)}{\pi n} \right\} \tag{1}$$

(d) Determine a closed-form expression for the autocorrelation sequence $r_{yy}[\ell]$ for the signal $y[n]$ below.

$$y[n] = (-1)^n x[n] = (-1)^n \left\{ \frac{\sin(\frac{\pi}{2}n)}{\pi n} \right\} \tag{2}$$

(e) Determine a closed-form expression for the autocorrelation sequence $r_{zz}[\ell]$ for the complex-valued signal $z[n]$ formed with $x[n]$ and $y[n]$ defined above as the real and imaginary parts, respectively, as defined below. *You must show all work and simplify as much as possible.*

$$z[n] = x[n] + jy[n] \tag{3}$$

(f) Plot $r_{zz}[\ell]$.

Problem 2.

For all parts of this problem, the reconstructed spectrum is computed according to the equation below:

$$Y_r(\omega) = \sum_{k=0}^{N-1} Y_N(k) \frac{\sin\left[\frac{N}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]}{N \sin\left[\frac{1}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]} e^{-j\frac{N-1}{2}\left(\omega - \frac{2\pi k}{N}\right)} \tag{4}$$

(a) Let $x[n]$ be a finite-length sinewave of length $L = 8$ and $h[n]$ be a discrete-time rectangular pulse of length $M = 4$ as defined below:

$$x[n] = e^{j\frac{\pi}{4}n}\{u[n] - u[n-8]\} \qquad\qquad h[n] = u[n] - u[n-4]$$

(i) With $X_N(k)$ computed as the 16-pt DFT of $x[n]$ and $H_N(k)$ computed as the 16-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is used in Eqn (1) with $N = 16$. Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

(ii) With $X_N(k)$ computed as the 12-pt DFT of $x[n]$ and $H_N(k)$ computed as the 12-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is used in Eqn (1) with $N = 12$. Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

(iii) With $X_N(k)$ computed as the 8-pt DFT of $x[n]$ and $H_N(k)$ computed as the 8-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is used in Eqn (1) with $N = 8$. Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$. You can approximate $\sin\left(\frac{\pi}{8}\right) \approx \frac{\pi}{8}$.

(b) Let $y[n]$ be a sine-window of length $L = 16$ as defined below. For all sub-parts of part (b), $Y_N(k)$ is computed as a 32-pt DFT of $y[n]$ and used in Eqn (1) with $N = 32$.

$$y[n] = \sin\left(\frac{\pi}{16}(n + 0.5)\right)\{u[n] - u[n-16]\}$$

(i) Write a closed-form expression for the resulting reconstructed spectrum $Y_r(\omega)$.

(ii) What is the numerical value of $Y_N(3)$? That is, what is the numerical value of the 32-pt DFT of $y[n]$ for the value $k = 3$? (Note that $y[n]$ is of length $L = 16$.)

(iii) What is the numerical value of $Y_N(7)$? That is, what is the numerical value of the 32-pt DFT of $y[n]$ for the value $k = 7$?

(iv) What is the numerical value of $Y_N(0)$? That is, what is the numerical value of the 32-pt DFT of $y[n]$ for the value $k = 0$? You can approximate $\sin\left(\frac{\pi}{32}\right) \approx \frac{\pi}{32}$.

Problem 3.

Consider the following complex-valued Vestigial Sideband (VSB) filter.

$$h[n] = e^{j\frac{\pi}{3}n} h_{LP}[n]$$

Let $H_{LP}(\omega)$ denote the DTFT (frequency response) of the filter $h_{LP}[n]$. $H_{LP}(\omega)$ is real-valued and symmetric. For $-\pi < \omega < \pi$, $H_{LP}(\omega)$ is mathematically described as

$$H_{LP}(\omega) = \begin{cases} 2, & |\omega| < \frac{\pi}{4} \\ 1 + \cos[6(|\omega| - \frac{\pi}{4})], & \frac{3\pi}{12} < |\omega| < \frac{5\pi}{12} \\ 0, & \frac{5\pi}{12} < |\omega| < \pi \end{cases}$$

Note that for later purposes in this problem, we can break up $h[n]$ into its real and imaginary parts using Euler's formula as $h[n] = h_R[n] + jh_I[n]$, where:

$$h_R[n] = Re\{h[n]\} = h_{LP}[n] \cos\left(\frac{\pi}{3}n\right) \qquad h_I[n] = Im\{h[n]\} = h_{LP}[n] \sin\left(\frac{\pi}{3}n\right)$$

(a) Plot the magnitude of the DTFT of $h[n]$, $H(\omega)$, over $-\pi < \omega < \pi$.

(b) Consider the following input signal

$$x[n] = 4 \frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n} \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$$

Plot the magnitude of the DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$.

(c) The signal in part (b) is run through the filter in part (a) to produce the output $y[n]$

$$y[n] = x[n] * h[n]$$

Plot the magnitude of the DTFT of $y[n]$, $Y(\omega)$, over $-\pi < \omega < \pi$.

(d) Consider the real part of the complex-valued filter $h[n]$. Using Euler's formula, we have that

$$h_R[n] = Re\{h[n]\} = h_{LP}[n] \cos\left(\frac{\pi}{3}n\right)$$

Plot the magnitude of the DTFT of $h_R[n]$, $H_R(\omega)$, over $-\pi < \omega < \pi$.

(e) Consider the real part of the complex-valued signal $y[n]$. Since the input $x[n]$ is real-valued, from part (d) we have:

$$y_R[n] = Re\{y[n]\} = x[n] * h_R[n]$$

Plot the magnitude of the DTFT of $y_R[n]$ over $-\pi < \omega < \pi$.

(f) Consider the real-valued signal $z[n]$ below, where $y_I[n] = Im\{y[n]\} = x[n] * h_I[n]$:

$$z[n] = y_R[n] \cos\left(\frac{\pi}{4}n\right) - y_I[n] \sin\left(\frac{\pi}{4}n\right)$$

Note that $z[n]$ is the real part of $y[n]e^{j\frac{\pi}{4}n}$. Plot the DTFT of $z[n]$ over $-\pi < \omega < \pi$.

(g) Draw a block diagram for a system for recovering the original signal $x[n]$ from $z[n]$. If you use a lowpass filter in your block diagram, you must specify both the passband edge and the stopband edge.

Problem 4.

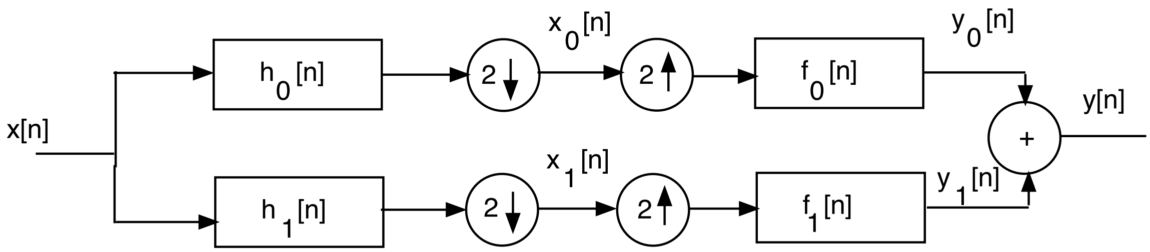
In the system below, the two analysis filters, $h_0[n]$ and $h_1[n]$, and the two synthesis filters, $f_0[n]$ and $f_1[n]$, form a Quadrature Mirror Filter (QMF). Specifically,

$$h_0[n] = \frac{2\beta\cos[(1+\beta)\pi(n+.5)/2]}{\pi[1-4\beta^2(n+.5)^2]} + \frac{\sin[(1-\beta)\pi(n+.5)/2]}{\pi[(n+.5)-4\beta^2(n+.5)^3]}, -\infty < n < \infty \text{ with } \beta = 0.5 \tag{5}$$

$$h_1[n] = (-1)^n h_0[n] \qquad f_0[n] = h_0[n] \qquad f_1[n] = -h_1[n]$$

The DTFT of the halfband filter $h_0[n]$ above may be expressed as follows:

$$H_0(\omega) = \begin{cases} e^{j\frac{\omega}{2}}, & |\omega| < \frac{\pi}{4} \\ e^{j\frac{\omega}{2}}\cos[(|\omega| - \frac{\pi}{4})], & \frac{\pi}{4} < |\omega| < \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} < |\omega| < \pi \end{cases}$$



Consider the input signal $x[n] = x_{rc}[n]\cos\left(\frac{\pi}{2}n\right)$ where the DTFT of $x_{rc}[n]$, denoted $X_{rc}(\omega)$, is given below:

$$X_{rc}(\omega) = \begin{cases} 2, & |\omega| < \frac{\pi}{4} \\ 1+\cos[4(|\omega| - \frac{\pi}{4})], & \frac{\pi}{4} < |\omega| < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| < \pi \end{cases}$$

THUS, the DTFT of the input signal is $X(\omega) = \frac{1}{2}X_{rc}\left(\omega - \frac{\pi}{2}\right) + \frac{1}{2}X_{rc}\left(\omega + \frac{\pi}{2}\right)$, where $X_{rc}(\omega)$ is defined above.

HINT: The solution to problem is greatly simplified if you exploit the fact that the DTFT of the input signal $x[n]$ is such that $X(\omega) = X(\omega - \pi)$.

(a) Plot the magnitude of the DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$.

(b) Plot the magnitude of the DTFT of $x_0[n]$, $X_0(\omega)$, over $-\pi < \omega < \pi$.

(c) Plot the magnitude of the DTFT of $x_1[n]$, $X_1(\omega)$, over $-\pi < \omega < \pi$.

(d) Plot the magnitude of the DTFT of $y_0[n]$, $Y_0(\omega)$, over $-\pi < \omega < \pi$.

(e) Plot the magnitude of the DTFT of $y_1[n]$, $Y_1(\omega)$, over $-\pi < \omega < \pi$.

(f) Plot the magnitude of the DTFT of the final output $y[n]$, $Y(\omega)$, over $-\pi < \omega < \pi$.