Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains FOUR problems.
All work should be done in the blue book provided.
Do not return this test sheet, just return your blue book.
Problem 1.

(a) Let \( x[n] \) and \( y[n] \) be real-valued sequences both of which are even-symmetric: \( x[n] = x[-n] \) and \( y[n] = y[-n] \). Under these conditions, prove that \( r_{xy}[\ell] = r_{yx}[\ell] \) for all \( \ell \).

(b) Express the autocorrelation sequence \( r_{zz}[\ell] \) for the complex-valued signal \( z[n] = x[n] + jy[n] \) where \( x[n] \) and \( y[n] \) are real-valued sequences, in terms of \( r_{xx}[\ell], r_{xy}[\ell], r_{yx}[\ell], \) and \( r_{yy}[\ell] \).

(c) Determine a closed-form expression for the autocorrelation sequence \( r_{xx}[\ell] \) for the signal \( x[n] \) below.

\[
x[n] = \left\{ \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n} \right\}
\]

(d) Determine a closed-form expression for the autocorrelation sequence \( r_{yy}[\ell] \) for the signal \( y[n] \) below.

\[
y[n] = (-1)^n x[n] = (-1)^n \left\{ \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n} \right\}
\]

(e) Determine a closed-form expression for the autocorrelation sequence \( r_{zz}[\ell] \) for the complex-valued signal \( z[n] \) formed with \( x[n] \) and \( y[n] \) defined above as the real and imaginary parts, respectively, as defined below. You must show all work and simplify as much as possible.

\[
z[n] = x[n] + jy[n]
\]

(f) Plot \( r_{zz}[\ell] \).
Problem 2.

For all parts of this problem, the reconstructed spectrum is computed according to the equation below:

\[ Y_r(\omega) = \sum_{k=0}^{N-1} Y_N(k) \frac{\sin \left[ \frac{N}{2} \left( \omega - \frac{2\pi k}{N} \right) \right]}{N \sin \left[ \frac{\omega}{2} \left( \omega - \frac{2\pi k}{N} \right) \right]} e^{-j \frac{N-1}{2} \left( \omega - \frac{2\pi k}{N} \right)} \]  \hspace{1cm} (4)

(a) Let \( x[n] \) be a finite-length sinewave of length \( L = 8 \) and \( h[n] \) be a discrete-time rectangular pulse of length \( M = 4 \) as defined below:

\[ x[n] = e^{j \frac{\pi}{8} n} \{ u[n] - u[n - 8] \} \quad \quad h[n] = u[n] - u[n - 4] \]

(i) With \( X_N(k) \) computed as the 16-pt DFT of \( x[n] \) and \( H_N(k) \) computed as the 16-pt DFT of \( h[n] \), the product \( Y_N(k) = X_N(k)H_N(k) \) is used in Eqn (1) with \( N = 16 \). Write a closed-form expression for the reconstructed spectrum \( Y_r(\omega) \).

(ii) With \( X_N(k) \) computed as the 12-pt DFT of \( x[n] \) and \( H_N(k) \) computed as the 12-pt DFT of \( h[n] \), the product \( Y_N(k) = X_N(k)H_N(k) \) is used in Eqn (1) with \( N = 12 \). Write a closed-form expression for the reconstructed spectrum \( Y_r(\omega) \).

(iii) With \( X_N(k) \) computed as the 8-pt DFT of \( x[n] \) and \( H_N(k) \) computed as the 8-pt DFT of \( h[n] \), the product \( Y_N(k) = X_N(k)H_N(k) \) is used in Eqn (1) with \( N = 8 \). Write a closed-form expression for the reconstructed spectrum \( Y_r(\omega) \). You can approximate \( \sin \left( \frac{\pi}{8} \right) \approx \frac{\pi}{8} \).

(b) Let \( y[n] \) be a sine-window of length \( L = 16 \) as defined below. For all sub-parts of part (b), \( Y_N(k) \) is computed as a 32-pt DFT of \( y[n] \) and used in Eqn (1) with \( N = 32 \).

\[ y[n] = \sin \left( \frac{\pi}{16}(n + 0.5) \right) \{ u[n] - u[n - 16] \} \]

(i) Write a closed-form expression for the resulting reconstructed spectrum \( Y_r(\omega) \).

(ii) What is the numerical value of \( Y_N(3) \)? That is, what is the numerical value of the 32-pt DFT of \( y[n] \) for the value \( k = 3 \)? (Note that \( y[n] \) is of length \( L = 16 \).)

(iii) What is the numerical value of \( Y_N(7) \)? That is, what is the numerical value of the 32-pt DFT of \( y[n] \) for the value \( k = 7 \)?

(iv) What is the numerical value of \( Y_N(0) \)? That is, what is the numerical value of the 32-pt DFT of \( y[n] \) for the value \( k = 0 \)? You can approximate \( \sin \left( \frac{\pi}{32} \right) \approx \frac{\pi}{32} \).
Consider the following complex-valued Vestigial Sideband (VSB) filter.

\[ h[n] = e^{j\frac{\pi}{3}n}h_{LP}[n] \]

Let \( H_{LP}(\omega) \) denote the DTFT (frequency response) of the filter \( h_{LP}[n] \). \( H_{LP}(\omega) \) is real-valued and symmetric. For \(-\pi < \omega < \pi\), \( H_{LP}(\omega) \) is mathematically described as

\[
H_{LP}(\omega) = \begin{cases} 
2, & |\omega| < \frac{\pi}{4} \\
1 + \cos[6(|\omega| - \frac{\pi}{4})], & \frac{3\pi}{12} < |\omega| < \frac{5\pi}{12} \\
0, & \frac{5\pi}{12} < |\omega| < \pi
\end{cases}
\]

Note that for later purposes in this problem, we can break up \( h[n] \) into its real and imaginary parts using Euler’s formula as

\[
h[n] = h_R[n] + jh_I[n] \]

(a) Plot the magnitude of the DTFT of \( h[n] \), \( H(\omega) \), over \(-\pi < \omega < \pi\).

(b) Consider the following input signal

\[ x[n] = 4 \sin \left( \frac{\pi}{3}n \right) \sin \left( \frac{\pi}{4}n \right) \]

Plot the magnitude of the DTFT of \( x[n] \), \( X(\omega) \), over \(-\pi < \omega < \pi\).

(c) The signal in part (b) is run through the filter in part (a) to produce the output \( y[n] \)

\[ y[n] = x[n] * h[n] \]

Plot the magnitude of the DTFT of \( y[n] \), \( Y(\omega) \), over \(-\pi < \omega < \pi\).

(d) Consider the real part of the complex-valued filter \( h[n] \). Using Euler’s formula, we have that

\[ h_R[n] = Re\{h[n]\} = h_{LP}[n] \cos \left( \frac{\pi}{3}n \right) \]

Plot the magnitude of the DTFT of \( h_R[n] \), \( H_R(\omega) \), over \(-\pi < \omega < \pi\).

(e) Consider the real part of the complex-valued signal \( y[n] \). Since the input \( x[n] \) is real-valued, from part (d) we have:

\[ y_R[n] = Re\{y[n]\} = x[n] * h_R[n] \]

Plot the magnitude of the DTFT of \( y_R[n] \) over \(-\pi < \omega < \pi\).

(f) Consider the real-valued signal \( z[n] \) below, where \( y_I[n] = Im\{y[n]\} = x[n] * h_I[n] \):

\[ z[n] = y_R[n] \cos \left( \frac{\pi}{4}n \right) - y_I[n] \sin \left( \frac{\pi}{4}n \right) \]

Note that \( z[n] \) is the real part of \( y[n]e^{j\frac{\pi}{4}n} \). Plot the DTFT of \( z[n] \) over \(-\pi < \omega < \pi\).

(g) Draw a block diagram for a system for recovering the original signal \( x[n] \) from \( z[n] \). If you use a lowpass filter in your block diagram, you must specify both the passband edge and the stopband edge.
Problem 4.

In the system below, the two analysis filters, \( h_0[n] \) and \( h_1[n] \), and the two synthesis filters, \( f_0[n] \) and \( f_1[n] \), form a Quadrature Mirror Filter (QMF). Specifically,

\[
h_0[n] = \frac{2\beta \cos[(1 + \beta)\pi(n + 0.5)/2]}{\pi[1 - 4\beta^2(n + 0.5)^2]} + \frac{\sin[(1 - \beta)\pi(n + 0.5)/2]}{\pi[(n + 0.5) - 4\beta^2(n + 0.5)^2]}, -\infty < n < \infty \quad \text{with} \quad \beta = 0.5
\]

\[
h_1[n] = (-1)^n h_0[n] \quad f_0[n] = h_0[n] \quad f_1[n] = -h_1[n]
\]

The DTFT of the halfband filter \( h_0[n] \) above may be expressed as follows:

\[
H_0(\omega) = \begin{cases} 
e^j\frac{\omega}{2}, & |\omega| < \frac{\pi}{4} \
e^j\frac{\omega}{2}\cos[(|\omega| - \frac{\pi}{4})], & \frac{\pi}{4} < |\omega| < \frac{3\pi}{4} \
e^j\frac{\omega}{2}, & \frac{3\pi}{4} < |\omega| < \pi \end{cases}
\]

Consider the input signal \( x[n] = x_{rc}[n]\cos\left(\frac{\pi}{2}n\right) \) where the DTFT of \( x_{rc}[n] \), denoted \( X_{rc}(\omega) \), is given below:

\[
X_{rc}(\omega) = \begin{cases} 2, & |\omega| < \frac{\pi}{4} \
1 + \cos[4(|\omega| - \frac{\pi}{4})], & \frac{\pi}{4} < |\omega| < \frac{\pi}{2} \
0, & \frac{\pi}{2} < |\omega| < \pi \end{cases}
\]

Thus, the DTFT of the input signal is \( X(\omega) = \frac{1}{2}X_{rc}\left(\omega - \frac{\pi}{2}\right) + \frac{1}{2}X_{rc}\left(\omega + \frac{\pi}{2}\right) \), where \( X_{rc}(\omega) \) is defined above.

Hint: The solution to problem is greatly simplified if you exploit the fact that the DTFT of the input signal \( x[n] \) is such that \( X(\omega) = X(\omega - \pi) \).

(a) Plot the magnitude of the DTFT of \( x[n] \), \( X(\omega) \), over \(-\pi < \omega < \pi\).

(b) Plot the magnitude of the DTFT of \( x_0[n] \), \( X_0(\omega) \), over \(-\pi < \omega < \pi\).

(c) Plot the magnitude of the DTFT of \( x_1[n] \), \( X_1(\omega) \), over \(-\pi < \omega < \pi\).

(d) Plot the magnitude of the DTFT of \( y_0[n] \), \( Y_0(\omega) \), over \(-\pi < \omega < \pi\).

(e) Plot the magnitude of the DTFT of \( y_1[n] \), \( Y_1(\omega) \), over \(-\pi < \omega < \pi\).

(f) Plot the magnitude of the DTFT of the final output \( y[n] \), \( Y(\omega) \), over \(-\pi < \omega < \pi\).