Digital Signal Processing I **Final Exam** **Fall 2009**
ECE538 **15 Dec.. 2009**

Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains **FIVE** problems.
All work should be done in the blue book provided.
Do **not** return this test sheet, just return your blue book.

Problem 1. Consider a finite-length sinewave of the form below where k_o is an integer in the range $0 \le k_o \le N-1$.

$$x[n] = e^{j2\pi \frac{k_o}{N} n}\{u[n] - u[n-N]\} \tag{1}$$

In addition, $h[n]$ is a causal FIR filter of length L, where $L < N$. In this problem $y[n]$ is the linear convolution of the causal sinewave of length N in Equation (1) with a causal FIR filter of length L, where $L < N$.

$$y[n] = x[n] * h[n]$$

(a) The region $0 \le n \le L-1$ corresponds to *partial overlap*. The convolution sum can be written as:

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad \textit{partial overlap: } 0 \le n \le L-1 \tag{2}$$

Determine the upper and lower limits in the convolution sum above for $0 \le n \le L-1$.

(b) The region $L \le n \le N-1$ corresponds to *full overlap*. The convolution sum is:

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad \textit{full overlap: } L \le n \le N-1 \tag{3}$$

(i) Determine the upper & lower limits in the convolution sum for $L \le n \le N-1$.

(ii) Substituting $x[n]$ in Eqn (1), show that for this range $y[n]$ simplifies to:

$$y[n] = H_N(k_o)e^{j2\pi \frac{k_o}{N} n} \quad \text{for} \quad L \le n \le N-1 \tag{4}$$

where $H_N(k)$ is the N-point DFT of $h[n]$ evaluated at $k = k_o$. To get the points, you must show all work and explain all details.

(c) The region $N \le n \le N+L-2$ corresponds to *partial overlap*. The convolution sum:

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad \textit{partial overlap: } N \le n \le N+L-2 \tag{5}$$

Determine the upper & lower limits in the convolution sum for $N \le n \le N+L-2$.

(d) Add the two regions of partial overlap at the beginning and end to form:

$$z[n] = y[n] + y[n+N] = \sum_{k=??}^{??} h[k]x[n-k] \quad \text{for: } 0 \le n \le L-1 \tag{6}$$

(i) Determine the upper and lower limits in the convolution sum above.

(ii) Substituting $x[n]$ in Eqn (1), show that for this range $z[n]$ simplifies to:

$$z[n] = y[n] + y[n+N] = H_N(k_o)e^{j2\pi \frac{k_o}{N} n} \quad \text{for } 0 \le n \le L-1 \tag{7}$$

where $H_N(k)$ is the N-point DFT of $h[n]$ evaluated at $k = k_o$ as defined previously. To get the points, you must show all work and explain all details.

(e) $y_N[n]$ is formed by computing $X_N(k)$ as an N-pt DFT of $x[n]$ in Eqn 1, $H_N(k)$ as an N-pt DFT of $h[n]$, and then $y_N[n]$ as the N-pt inverse DFT of $Y_N(k) = X_N(k)H_N(k)$. Write a closed-form expression for $y_N[n]$. Is $z[n] = y_N[n]$?

Problem 2.

For all parts of this problem, the reconstructed spectrum is computed according to Equation 1 below:

$$X_r(\omega) = \sum_{k=0}^{N-1} X_N(k) \frac{\sin\left[\frac{N}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]}{N \sin\left[\frac{1}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]} e^{-j\frac{N-1}{2}\left(\omega - \frac{2\pi k}{N}\right)} \tag{8}$$

(a) Let $x[n]$ be a discrete-time rectangular pulse of length $L = 12$ as defined below:

$$x[n] = \{-1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1\}$$

(i) $X_N(k)$ is computed as a 16-point DFT of $x[n]$ and used in Eqn (1) with $N = 16$. Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.

(ii) $X_N(k)$ is computed as a 12-point DFT of $x[n]$ and used in Eqn (1) with $N = 12$. Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.

(iii) $X_N(k)$ is computed as an 8-point DFT of $x[n]$ and used in Eqn (1) with $N = 8$. That is, $X_N(k)$ is obtained by sampling the DTFT of $x[n]$ at 8 equi-spaced frequencies between 0 and 2π. Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.

(b) Let $x[n]$ be a discrete-time sinewave of length $L = 12$ as defined below. For all subparts of part (b), $X_N(k)$ is computed as a 12-pt DFT of $x[n]$ and used in Eqn (1) with $N = 12$.

$$x[n] = \cos\left(\frac{\pi}{3}n\right)\{u[n] - u[n-12]\}$$

(i) Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.

(ii) What is the numerical value of $X_r(\frac{\pi}{3})$? The answer is a number and you do not need a calculator to determine the value; this also applies to the next 2 parts.

(iii) What is the numerical value of $X_r(\frac{5\pi}{3})$?

(iv) What is the numerical value of $X_r(\frac{\pi}{2})$?

Problem 3. This problem is about Vestigial Sideband (VSB) Modulation where we transmit a small part of the negative portion of the spectrum along with the positive frequency portion of the spectrum. *Note: all signals and filters in this problem have zero-phase in the frequency domain.*

(a) Consider the following complex-valued filter.

$$h[n] = 16\, e^{j\frac{7\pi}{16}n} \left\{ \frac{\sin\left(\frac{2\pi}{16}n\right)}{\pi n}\ \frac{\sin\left(\frac{7\pi}{16}n\right)}{\pi n} \right\}$$

Plot the DTFT of $h[n]$, $H(\omega)$, over $-\pi < \omega < \pi$.

(b) For illustrative purposes, consider the following simple input signal

$$x[n] = \frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n}$$

Plot the DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$.

(c) The signal in part (b) is run through the filter in part (a) to produce the output $y[n]$

$$y[n] = x[n] * h[n]$$

Plot the DTFT of $y[n]$, $Y(\omega)$, over $-\pi < \omega < \pi$.

(d) The real-part of the complex-valued signal $y[n]$ may be expressed as

$$y_R[n] = Re\{y[n]\} = \frac{1}{2}\{y[n] + y^*[n]\}$$

Plot the DTFT of the complex-conjugate signal $y^*[n]$ over $-\pi < \omega < \pi$.

(e) Sum your respective answers to parts (c) and (d), and divide by 2, to form the DTFT of $y_R[n]$, denoted $Y_R(\omega)$. Plot $Y_R(\omega)$ over $-\pi < \omega < \pi$.

(f) Is your answer to part (e) equal to the DTFT of the original signal $x[n]$? That is, is $y_R[n] = x[n]$? Why or why not? Explain your answer.

(g) The output of the filter in part (a) includes some of the frequency content of $x[n]$ in the band $-\frac{\pi}{8} < \omega < 0$ in addition to the positive frequency portion of the signal in $0 < \omega < \pi$. Consider a more general case where the output of the filter $h[n]$ includes some of the frequency content of $x[n]$ in the band $-\Delta < \omega < 0$, in addition to the positive frequency portion of the signal in $0 < \omega < \pi$. In addition to the constraints $H(\omega) = 2$ over $\Delta < \omega < \pi$, and $H(\omega) = 0$ over $-\pi < \omega < -\Delta$, what condition does $H(\omega)$ have to satisfy over $-\Delta < \omega < \Delta$ in order that the real part of the output $y[n] = x[n] * h[n]$ be equal to $x[n]$?

Problem 4.

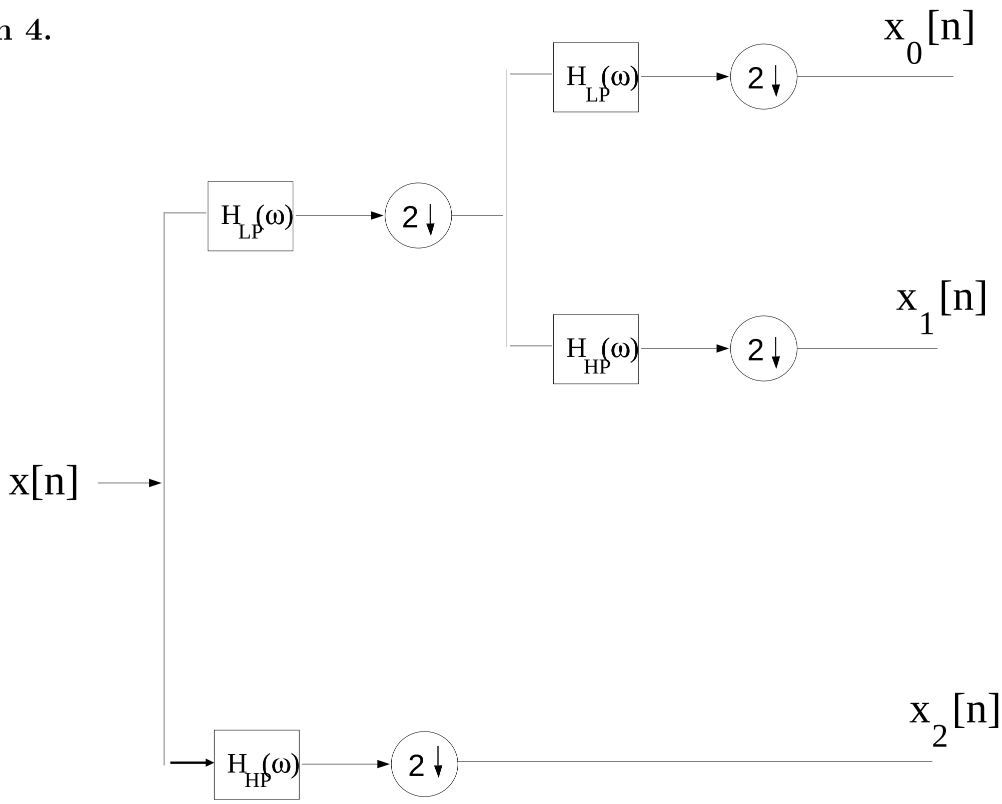

Figure 1(a). Analysis Section of Two-Stage Tree-Structured Filter Bank.

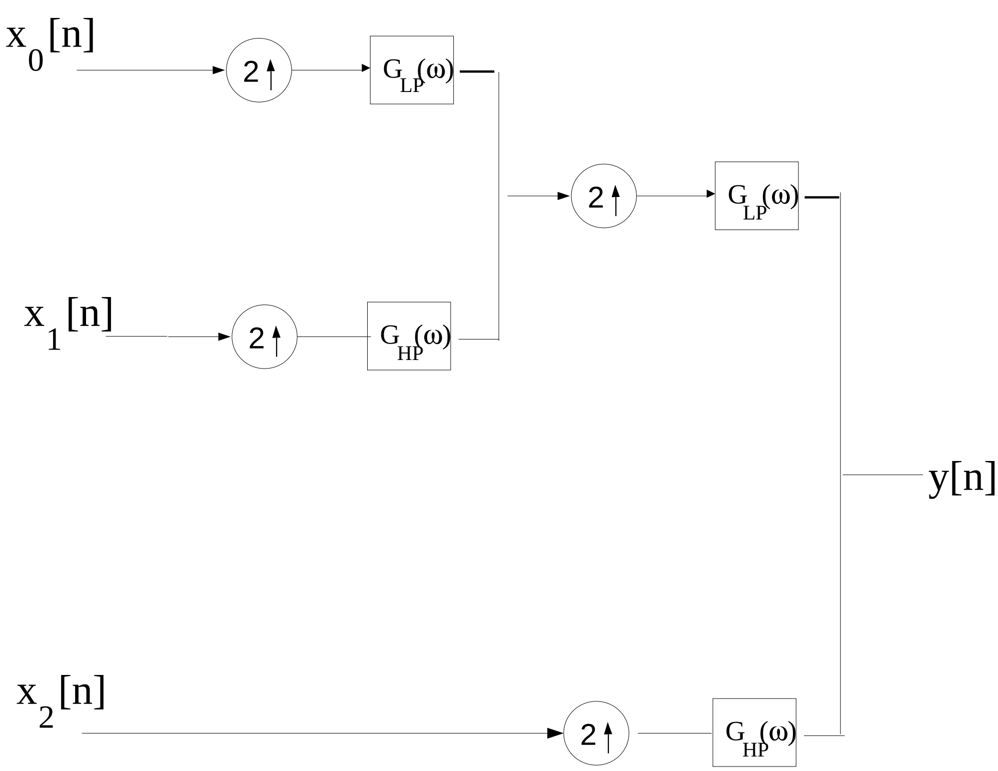

Figure 1(b). Synthesis Section of Two-Stage Tree-Structured Filter Bank.

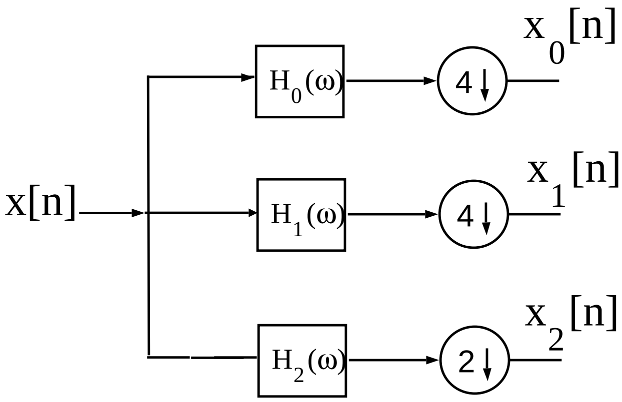

Figure 2(a). Analysis Filter Bank.

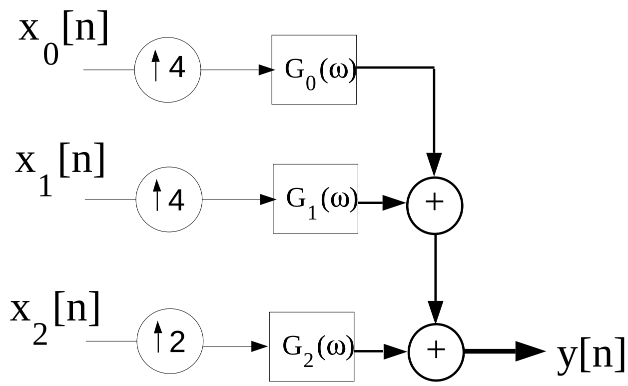

Figure 2(b). Synthesis Filter Bank.

This problem is about synthesizing an M=3 channel nonuniform PR filter bank from a two stage tree-structured PR filter bank.

$$h_{LP}[n] = \{1, 1\} = u[n] - n[n-2]$$

$$h_{HP}[n] = (-1)^n h_{LP}[n]$$

Also, $g_{LP}[n] = h_{LP}[n]$ and $g_{HP}[n] = -h_{HP}[n]$. The combination of the analysis filter pair, $\{H_{LP}(\omega), H_{HP}(\omega)\}$, and synthesis filter pair $\{G_{LP}(\omega), G_{HP}(\omega)\}$, form a two-channel PR filter bank.

(a) Use Noble's First Identity to determine the analysis filters $H_m(\omega)$, $m = 0, 1, 2$, so that the system in Fig. 2(a) is equivalent to the system in Fig. 1(a).

  (i) Write a time-domain expression for EACH of the three filters $h_m[n]$, $m = 0, 1, 2$.

  (ii) Plot the frequency response for each of the three filters $H_m(\omega)$, $m = 0, 1, 2$.

(b) Next, use Noble's Second Identity to express each synthesis filter, $G_m(\omega)$, $m = 0, 1, 2$, in terms of $G_{LP}(\omega)$ and $G_{HP}(\omega)$, so that the system in Fig. 2(b) is equivalent to the system in Fig. 1(b).

(c) Plot the magnitude of $F(\omega) = \sum_{i=0}^{2} H_i(\omega) G_i(\omega)$ over $-\pi < \omega < \pi$.

Problem 5.

Consider the autoregressive AR(2) process generated via the difference equation

$$x[n] = x[n-1] - \frac{1}{2}x[n-2] + \nu[n]$$

where $\nu[n]$ is a stationary white noise process with variance $\sigma_w^2 = 5/2 = 2.5$. The first three autocorrelation values ($r_{xx}[m] = E\{x[n]x[n-m]\}$) for this AR process have the numerical values indicated below:

$$r_{xx}[0] = 6 \qquad r_{xx}[1] = 4 \qquad r_{xx}[2] = 1$$

(a) Determine the value of $r_{xx}[3]$.

(b) Determine a simple, closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m]$:

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m]e^{-jm\omega}$$

(c) Consider the first-order predictor

$$\hat{x}[n] = -a_1(1)x[n-1]$$

Determine the numerical value of the optimum predictor coefficient $a_1(1)$ and the corresponding minimum mean-square error.

(d) Consider the third-order predictor

$$\hat{x}[n] = -a_3(1)x[n-1] - a_3(2)x[n-2] - a_3(3)x[n-3]$$

Determine the numerical values of the optimum predictor coefficients $a_3(1)$, $a_3(2)$, and $a_3(3)$ and the corresponding minimum mean-square error.