Digital Signal Processing I **Final Exam** **Fall 2008**
ECE538 **17 Dec.. 2008**

Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains **FIVE** problems.
All work should be done on blank 8.5" x 11" white sheets of paper (NOT provided).
Do **not** return this test sheet, just return your answer sheets.

Problem 1. [20 pts]

You are given the matrix inversion lemma:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

A symmetric-Toeplitz matrix is formed from the first three autocorrelation values of an AR process as $r_{xx}[0] = 1/3$, $r_{xx}[1] = 1/6$, and $r_{xx}[2] = 1/12$ as

$$\mathbf{R}_3 = \begin{bmatrix} 1/3 & 1/6 & 1/12 \\ 1/6 & 1/3 & 1/6 \\ 1/12 & 1/6 & 1/3 \end{bmatrix}$$

(a) Use the matrix inversion lemma to compute the inverse of $\mathbf{R}_3$ in terms of the inverse of

$$\mathbf{R}_2 = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

$$\mathbf{R}_2^{-1} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

(b) Use the matrix inversion lemma to compute the solution

$$\mathbf{a}_3 = -\mathbf{R}_3^{-1}\mathbf{r}_3 \quad \text{where: } \mathbf{r}_3 = \begin{bmatrix} 1/6 \\ 1/12 \\ 1/24 \end{bmatrix}$$

in terms of

$$\mathbf{a}_2 = -\mathbf{R}_2^{-1}\mathbf{r}_2 = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \quad \text{where: } \mathbf{r}_2 = \begin{bmatrix} 1/6 \\ 1/12 \end{bmatrix}$$

(c) What is the model order p of the AR process?

Problem 2. [20 points]

A signal $x[n]$ of length 12 is broken up into two overlapping blocks of length 8, denoted $x_1[n]$ and $x_2[n]$, respectively, for the purposes of filtering with $h[n] = \{-1, 2, -1\}$ of length $M = 3$ via the overlap-SAVE method. The last $M - 1 = 2$ points of $x_1[n]$ are the first two points of $x_2[n]$. The first $M - 1 = 2$ zeros in $x_1[n]$ were added to get the overlap-save method started, as done in Fig. 7.3.1 in the text.

$$x_1[n] = \{0, 0, 1, 1, 1, 1, 1, 1\}$$

and

$$x_2[n] = \{1, 1, -1, -1, -1, -1, -1, -1\}$$

(a) $y_1[n]$ is formed by computing $X_1(k)$ as an 8-pt DFT of $x_1[n]$, $H(k)$ as an 8-pt DFT of $h[n]$, and then $y_1[n]$ as the 8-pt inverse DFT of $Y_1(k) = X_1(k)H(k)$. Write out the values of $y_1[n]$ in sequence form (similar to how $x_1[n]$ and $x_2[n]$ are written out above.)

(b) $y_2[n]$ is formed by computing $X_2(k)$ as an 8-pt DFT of $x_2[n]$, $H(k)$ as an 8-pt DFT of $h[n]$, and then $y_2[n]$ as the 8-pt inverse DFT of $Y_2(k) = X_2(k)H(k)$. Write out the 8 values of $y_2[n]$ in sequence form.

(c) Show how $y_1[n]$ and $y_2[n]$ are combined to form the full linear convolution $y[n] = x[n] * h[n]$, via the overlap-save method.

Problem 3. [20 points]

A second-order digital filter is to be designed from an analog filter having two poles in the s-plane at $-0.2 + 0.4j$ and $-0.2 - 0.4j$ and two zeros at $j\sqrt{3}$ and $-j\sqrt{3}$, via the bilinear transformation method characterized by the mapping

$$s = \frac{z-1}{z+1}$$

Note that $-0.2 + 0.4j = -(1/5) + j(2/5)$ and $j\sqrt{3} = j\tan(\pi/3)$

(a) Is the resulting digital filter (BIBO) stable? Briefly explain why or why not.

(b) Denote the frequency response of the resulting digital filter as $H(\omega)$ (the DTFT of its impulse response). You are given that in the range $0 < \omega < \pi$, there is only one value of ω for which $H(\omega) = 0$. Determine that value of ω.

(c) Draw a pole-zero diagram for the resulting **digital** filter. Give the exact locations of the poles and zeros of the digital filter in the z-plane.

(d) Plot the magnitude of the DTFT of the resulting digital filter, $|H(\omega)|$, over $-\pi < \omega < \pi$. You are given that $H(0) = 6$. Be sure to indicate any frequency for which $|H(\omega)| = 0$. Also, specifically note the numerical value of $|H(\omega)|$ for $\omega = \frac{\pi}{2}$ and $\omega = \pi$.

(e) Determine the difference equation for the resulting digital filter.

Problem 4. [20 points]

(a) Consider a causal FIR filter of length $M = 14$ with impulse response

$$h[n] = u[n] - u[n-14]$$

Provide a **closed-form** expression for the 16-pt DFT of $h[n]$, denoted $H_{16}(k)$, as a function of k. Simplify as much as possible. You are given the following four values:

$$H_{16}(0) = 14 \quad H_{16}(4) = 1 - j \quad H_{16}(8) = 0 \quad H_{16}(12) = 1 + j$$

(b) Consider the sequence $x[n]$ of length $L = 16$ below, equal to a sum of several finite-length sinewaves.

$$x[n] = 1 + 2\cos\left(\frac{\pi}{2}n\right) + \cos(\pi n), \quad n = 0, 1, ..., 15.$$

$y_{16}[n]$ is formed by computing $X_{16}(k)$ as a 16-pt DFT of $x[n]$, $H_{16}(k)$ as a 16-pt DFT of $h[n]$, and then $y_{16}[n]$ as the 16-pt inverse DFT of $Y_{16}(k) = X_{16}(k)H_{16}(k)$. Express the result $y_{16}[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

(c) For the remaining parts of this problem, $h[n]$ is now defined as the causal FIR filter of length $M = 14$ below.

$$h[n] = (-1)^n\{u[n] - u[n-14]\}$$

Provide a **closed-form** expression for the 16-pt DFT of $h[n]$, denoted $H_{16}(k)$, as a function of k. Simplify as much as possible.

(d) Consider again the sequence $x[n]$ of length $L = 16$ below.

$$x[n] = 1 + 2\cos\left(\frac{\pi}{2}n\right) + \cos(\pi n), \quad n = 0, 1, ..., 15.$$

$y_{16}[n]$ is formed by computing $X_{16}(k)$ as an 16-pt DFT of $x[n]$, $H_{16}(k)$ as a 16-pt DFT of $h[n]$, and then $y_{16}[n]$ as the 16-pt inverse DFT of $Y_{16}(k) = X_{16}(k)H_{16}(k)$. Express the result $y_{16}[n]$ as a weighted sum of finite-length sinewaves.

(e) Consider the sequence $p[n]$ of length $L = 16$ below.

$$p[n] = 4\delta[n] + 4\delta[n-4] + 4\delta[n-8] + 4\delta[n-12]$$

Is $p[n] = x[n]$? If they are equal, provide an explanation as to why they are equal.

Problem 5. [20 points]

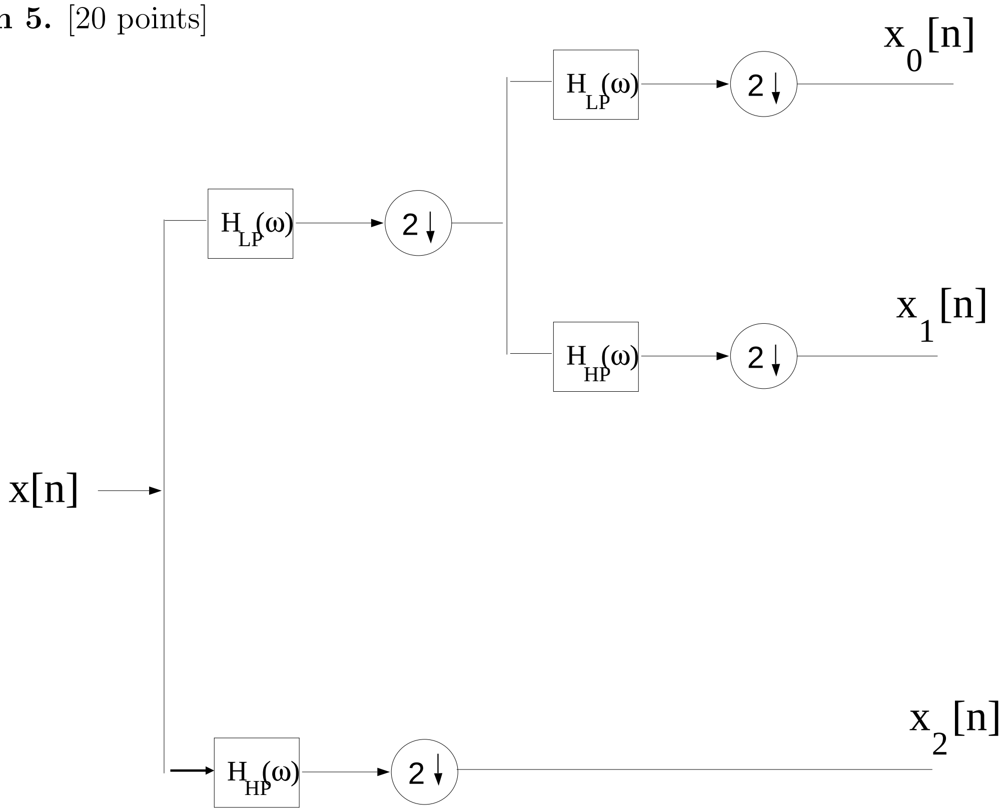

Figure 1(a). Analysis Section of Two-Stage Tree-Structured Filter Bank.

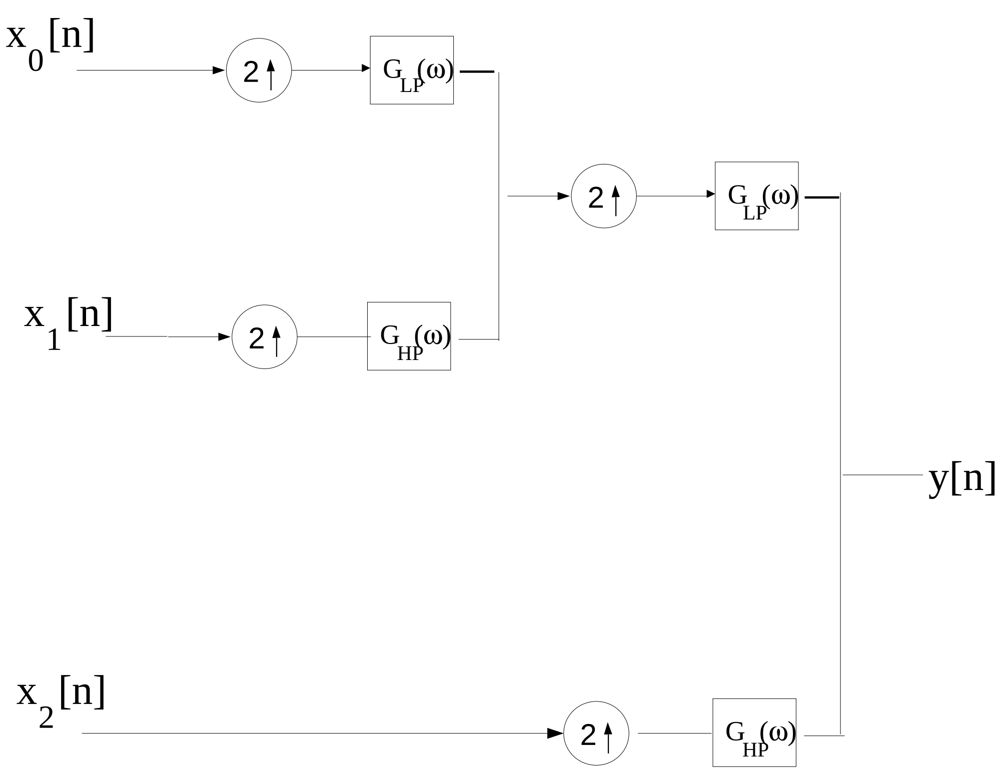

Figure 1(b). Synthesis Section of Two-Stage Tree-Structured Filter Bank.

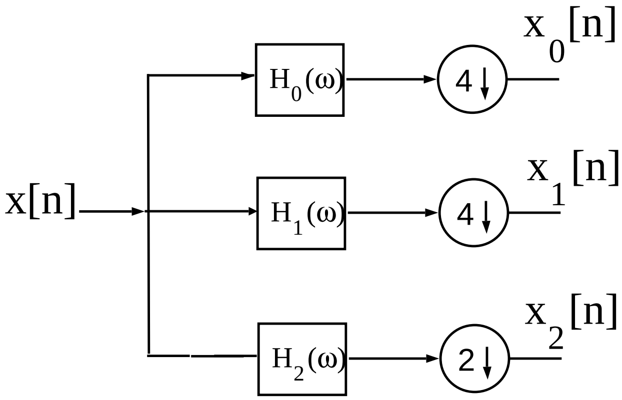

Figure 2(a). Analysis Filter Bank, $M = 4$.

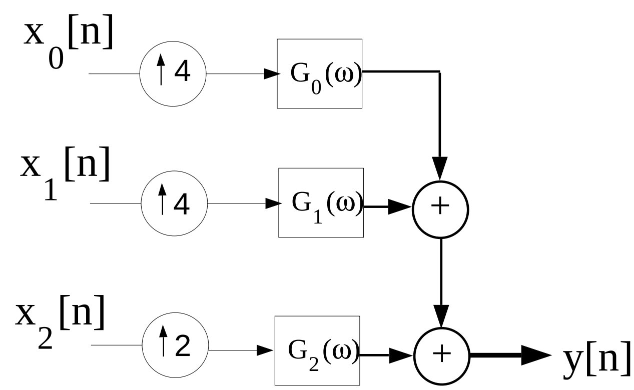

Figure 2(b). Synthesis Filter Bank, $M = 4$.

This problem is about synthesizing an M=3 channel nonuniform PR filter bank from a two stage tree-structured PR filter bank.

$$h_{LP}[n] = \frac{\sin\left(\frac{\pi}{2}(n+0.5)\right)}{\frac{\pi}{2}(n+0.5)}$$

$$h_{HP}[n] = (-1)^n h_{LP}[n]$$

Also, $g_{LP}[n] = h_{LP}[n]$ and $g_{HP}[n] = -h_{HP}[n]$. The combination of the analysis filter pair, $\{H_{LP}(\omega), H_{HP}(\omega)\}$, and synthesis filter pair $\{G_{LP}(\omega), G_{HP}(\omega)\}$, form a two-channel PR filter bank.

(a) Use Noble's First Identity to determine the analysis filters $H_m(\omega)$, $m = 0, 1, 2$, so that the system in Fig. 2(a) is equivalent to the system in Fig. 1(a).

  (i) Plot the frequency response for each of the three filters $H_m(\omega)$, $m = 0, 1, 2$. You can put all three plots on the same graph to save space.

  (ii) Write a time-domain expression for EACH of the three filters $h_m[n]$, $m = 0, 1, 2$.

(b) Next, use Noble's Second Identity to express each synthesis filter, $G_m(\omega)$, $m = 0, 1, 2$, in terms of $G_{LP}(\omega)$ and $G_{HP}(\omega)$.

(c) Plot $F(\omega) = \sum_{i=0}^{2} H_i(\omega) G_i(\omega)$ over $-\pi < \omega < \pi$.